

Supply and Demand in a Two-Sector Matching Model*

Paweł Gola[†]

May 2016

Abstract

This paper explores an extension of Becker's (1973) matching model to two sectors and two skill dimensions. I provide comparative statics for changes in the marginal distribution of skill and in skill interdependence, as well as in the distribution of firms' productivity. Vertical differentiation of workers proves to be of crucial importance for sorting, and also, as a result, for shock transmission. Further, as each sector uses a different dimension of skill, increases in skill interdependence imply a fall in the overall supply of talent. In the symmetric case, at least, this results in lower output and higher wage inequality.

Keywords: two-sector matching, vertical differentiation, talent supply, wage inequality.

JEL codes: C78, D31, J24, J31

*The code used to produce the figures and tables can be found at <http://www.pawelgola.com/research/>. I am very grateful to my doctoral advisor, Ian Jewitt, for his support and guidance. I would like to thank Peter Eso for feedback and encouragement, as well as Matej Bajgar, Claudia Herresthal, Ed Hopkins, Jashua Lanier, Karan Nagpal, Bartosz Redlicki, Federico Torracchi and Łukasz Woźny for detailed comments on an earlier draft. Finally, I would also like to thank Paweł Dziewulski, Jan Eeckhout, Matt Elliott, Aytak Erdil, Ben Golub, Meg Meyer, Climent Quintana-Domeque, Larry Samuelson, Coen Teulings and seminar participants at Oxford, Cambridge, Ce² 2014, WIEM 2014 and RES Conference 2015 for discussions and suggestions. Most of this work was carried out during my doctoral studies at the University of Oxford.

[†]Emmanuel College, University of Cambridge. E-mail: pg454@cam.ac.uk

1 Introduction

Labour market sorting shapes the future of the economy: the career choices of today's workers influence which sectors and countries will develop fastest in the next decades. Despite this, its determinants, as well as some of its consequences, are not very well understood. Think of the financial and production sectors and consider the following question: would the financial sector contract if the entire population became less skilled in banking? The answer is somewhat counter-intuitive and hinges precisely on the impact this change has on sorting. In this paper, I show that the equilibrium supply of talent in finance, i.e. the number of talented workers finance is able to attract, depends not just on the level of banking skill, but also on its spread: that is, the difference in productivity between high and low talent bankers. Subsequently, I explore the implications of this finding.

This paper explores a natural extension of Becker's (1973) frictionless matching model to two sectors and two skill dimensions, showing that this extension produces interesting comparative statics. There is a continuum of workers, each of them endowed with a bivariate vector of skills, as well as a continuum of firms grouped into two sectors. The first of workers' skills – x_1 – is used in finance and the second – x_2 – in manufacturing, which captures the idea that different sectors require different skillsets. The model allows for any degree of interdependence between these two skill dimensions. Every firm in sector $i = 1, 2$ is endowed with a scalar productivity z_i . A match between a single agent and a single firm produces surplus, according to a sector-specific surplus function, which always at least covers the reservation payoffs of both the worker and the firm. In line with the matching literature, I require that the surplus functions are increasing and supermodular. The equilibrium concept used in this paper is the stable matching.

In this introduction, I focus on two simple examples in order to present the basic results and intuition. The examples differ in the type of shock that hits the economy, but share the initial, pre-shock specification of the financial and production sectors. The two sectors are symmetric, the surplus in each is given by $(x_i + 1)(z_i + 1)$ and there are as many agents as firms. Firms' productivity in each industry is standard uniformly distributed, and the joint distribution of skill is defined on $[0, 1]^2$ and given by $x_1 x_2$ – banking and manufacturing skills are, therefore, independent.

First, consider a shock to the surplus produced in banking. To fix ideas, suppose that finance becomes more regulated, perhaps in response to the financial crisis. This negatively affects productivity in banking. Specifically, the surplus

Sector	Type of effect	% change in		
		surplus	wage variance	wage range
Finance	short run	-0.43	11.09	12.00
	long run	0.15	-0.11	2.77
Production	long run	-0.54	9.25	0.86

Table 1: The effects of the negative surplus shock.

produced in the financial sector changes to $(x_1 + 1 - 0.15(1 - x_1)^4)(z_1 + 1)$. This functional form implies that surplus falls for all bankers, but less so for highly talented ones.¹

Let us compare the impact of the shock in the short and long run. In the short run, sorting is fixed and, therefore, only the financial sector is affected. Unsurprisingly, as the shock is negative, finance contracts (by 0.43%). Additionally, wage variance and wage range increase in finance, because the highly skilled bankers, being less affected by regulation, become relatively more valuable. In the long run, however – that is, after the assignment of agents to sectors has responded to the shock – the results are strikingly different. Here, the financial sector expands by 0.15%, bankers’ wage variance falls and although wage range still increases, the size of the effect is much smaller than in the short run (2.77% compared to 12%). This would suggest that most of the negative consequences of the shock in the financial sector are exported to the production sector. And indeed, the production sector does contract (by 0.54%) and experiences an increase in both wage variance and wage range. All these effects are summarised in Table 1.²

In order to understand what causes this ‘export’ of negative consequences, note that from the point of view of any bank, the shock increases the difference in the surplus produced by workers of any two talent levels. This is equivalent to an increase in the *spread of surplus* in the sense of Bickel and Lehmann (1979) and implies that bankers’ become more *vertically differentiated*. Keeping prices constant, talent becomes underpaid in finance and all matched banks demand

¹There are two reasons why highly skilled bankers are likely to be less affected by regulation. Firstly, there is a widely-held belief, supported both by empirical findings (Lucca, Seru, and Trebbi, 2014) and economic theory (Bond and Glode, 2014), that bankers are on average more talented than their regulators and are, therefore, able to circumvent regulation. By the same logic, the more talented bankers should be better at finding workarounds. Alternatively, the externality produced in banking pre-regulation could have been greater – conditional on match – for less skilled agents (perhaps because they were substituting skill with excessive risk).

²Figures 1 and 2 in the Online Appendix depict the short and long run impact of both types of shock on wage distribution in each sector.

workers of higher talent than previously. Therefore, the *demand for talent* shifts up at the intensive margin. The extensive margin (i.e. the measure of banks demanding any workers at all) in principle depends also on the level of surplus produced. However, as jobs are weakly scarce in this example, the size of each sector is fixed and the extensive margin cannot change. Hence, talent demand shifts up overall, which results in an increase in the *supply of talent* in finance – that is, in the number of agents with banking talent of at least t (for all talent levels t). As a consequence, less talent is supplied to the production sector.

This has a positive effect on the output produced in finance – in this example the positive talent supply effect dominates the negative short run effect. In the production sector only the negative talent supply effect is present: hence, manufacturing contracts. In other words, it is the increase in vertical differentiation that enables finance to export at least part of the shock’s negative output effect to the production sector. If, instead, the shock decreased vertical differentiation, it would have a negative effect on the supply of talent in finance, which would then magnify the – also negative – short run effect.

The situation is slightly more complicated in the case of wage inequality, where the exact results depend on the measure of inequality. The general insight, however, is even more striking: here, the increase in bankers’ vertical differentiation has a self-mitigating effect on wage inequality in finance. Even though the direct effect is positive, the resulting change in talent supply exports at least part of that increase to the production sector. The mechanism is simple: the fall in the supply of manufacturing talent increases its relative price and raises the spread of wages in the production sector. This causes the rise in wage range, and contributes to the increase in variance.³ In finance, the spread of wage does not necessarily grow, as the greater supply of talent mitigates the positive short run effect of the rise in vertical differentiation.⁴ Nevertheless, wage range does go up, but less than in the short run. The change in wage variance is ambiguous.

Let us turn our attention to a shock that affects *the overall supply of talent*. For example, suppose that – ignoring its effect on productivity – the regulation of the financial system has changed the skillset needed to succeed in finance, so that banking skills became more correlated with manufacturing skills. Specifically, the post-shock joint distribution of skill is given by $x_1x_2[1 + 0.75(1 - x_1)(1 - x_2)]$ – i.e.

³In general, variance does not have to increase, because the associated change in talent distribution can, in some cases, work in the opposite direction. However, variance definitely raises for the incumbents, i.e. agents who work in manufacturing both prior and after the shock.

⁴ Bickel and Lehmann (1979) spread is a partial order. Hence, the spread of wage might change and yet neither increase nor decrease.

by the FGM copula with $\alpha = 0.5$ – which implies a correlation coefficient of 0.25. In both sectors, this results in a 1.27% fall in output, as well as in an increase in wage variance (by 16.14%) and wage range (by 1.87%). To see why, note that an increase in interdependence implies a fall in the number of agents who are highly talented in at least one dimension (irrespective of how ‘high’ talent is defined). As each sector uses a single dimension of skill, this is equivalent to a fall in *the overall supply of talent*.⁵ Furthermore, because of symmetry, a fall in overall supply of talent decreases the sectoral supply of talent in each industry – and the effects on output, range and variance follow from the considerations above.

The examples show that extending Becker’s model to two sectors can introduce striking new comparative statics, as well as introduce a potentially important transmission of shocks between different sectors of the economy. The aim of this paper is to display the forces at work in the context of a general model. It is structured as follows. The next section reviews the related literature and places my main contributions into it. Section 2 develops the model, and characterises the unique assignment of agents to sectors. Sections 3 and 4 provide comparative statics results for changes in workers’ vertical differentiation. In Section 3 these are caused by shifts in the marginal distribution of skills, and in Section 4 by improvements in the distribution of firms. Section 5 derives comparative statics for shocks to the overall supply of skill, captured by changes in the copula of the skill distribution. Section 6 concludes. All proofs can be found in the Appendix. Additional results for the case of abundant jobs are provided in the Online Appendix.

1.1 Related Literature

This paper builds on the work of Becker (1973), Sattinger (1979) and Roy (1951), combining their approaches towards matching and self-selection, respectively. My model nests Sattinger-like, one sector matching models and Roy-like, two sector comparative advantage models within one framework. The sectors in Roy (1951) can be interpreted as Becker-like matching markets with homogenous and abundant firms, implying that companies have no market power and workers earn the entire surplus. In Roy’s model, therefore, sorting depends only on surplus’ levels,

⁵To see why this is the case, consider two pairs of workers – the first pair is described by skill vectors $\{(x_1^H, x_2^L), (x_1^L, x_2^H)\}$ and the second by $\{(x_1^H, x_2^H), (x_1^L, x_2^L)\}$, where $x_i^H > x_i^L$. The first worker in the latter pair is highly skilled in both dimensions and, therefore, one of those two high skills will not be used. This does not occur for the former pair of workers, enabling them to produce more output.

not its spread. The introduction of firm heterogeneity gives firms' some market power and is the reason why the vertical differentiation of workers matters for talent supply. Compared to Becker (1973) and Sattinger (1979), the addition of another sector allows the study of interactions between two matching markets, as well as the determinants of sectoral talent supply.

There are a number of papers that provide comparative statics results for the standard, one-sector differential rents model (e.g. Costrell and Loury, 2004; Gabaix and Landier, 2008; Tervio, 2008).⁶ These results capture only the short run effect of the shocks, as the within-sector distribution of skill is fixed. As noted in Costrell and Loury (2004), this is a serious limitation – which is addressed directly in this paper. In particular, I show that the two types of shocks which are of particular importance in this literature (i.e. multiplicative surplus shocks and first order stochastic dominance improvements in the distribution of firms' productivity) result in a greater supply of skill in the affected sector, increasing, as they do, both the levels and the spread of surplus (Sections 3 and 4). This undoes at least part of their positive short run effect on wage levels and inequality. In fact, with scarce jobs, wages will certainly fall for the least talented agents in the industry hit by the shock.

My model is a multivariate matching problem. Most of the literature in this area is focused on marriage markets (Anderson, 2003; Chiappori, Oreffice, and Quintana-Domeque, 2011, 2012); however, a recent paper by Lindenlaub (2014) does investigate multivariate matching in labour markets. Lindenlaub defines positive and assortative matching in a general setting with bivariate skills and skill-demands, and provides sufficient conditions for its existence. However, the model is solved and comparative statics are provided only for the very special quadratic-Gaussian case.⁷ The comparative statics results focus on technological change, modelled as a multiplicative surplus shock, and only the knife-edge case of no unmatched firms and workers is considered. My paper studies the determinants of talent supply more generally, including changes to the distribution of skills, and all my results hold for general surplus and distribution functions. On the other hand, my model does not allow for jobs that require two types of skills – however, given Lindenlaub's finding that the demand for skills is strongly negatively correlated

⁶See Sattinger (1993) for an overview of the different types of assignment models.

⁷The quadratic surplus function coupled with normal skill distribution implies a surplus function that is not monotonic in agents' skills, so that agents with extremely high and extremely low skills produce the same surplus and earn the same wages. Adding non-interaction skill terms does not resolve this problem in a satisfactory manner, as evidenced by the fact that the surplus function estimated in Lindenlaub (2014) (Table 8) is non-monotonic.

in the US, this does not seem restrictive.

There exists a small, but quickly growing literature on multisector matching. The models in McCann, Shi, Siow, and Wolthoff (2015) and Grossman, Helpman, and Kircher (2013), however, differ substantially from the one presented here, both focusing on one-to-many rather than one-to-one matching. McCann et al. (2015) have a complicated model, with three markets and schooling. This comes at the cost of using specific functional forms and not providing comparative statics results. Meanwhile, Grossman et al. (2013) focus on the impact of trade liberalisation, rather than changes in skill and productivity distributions. Their skills are one-dimensional and they restrict attention to cases where re-sorting happens at the extensive margin only. The model in Dupuy (2015) is quite similar to the one presented here (albeit less general), as it is a differential rents matching model with two-dimensional ability.⁸ Dupuy proves the existence (but not uniqueness) of an equilibrium and then proceeds to study the impact of multiplicative shocks on self-selection and inequality. However, unlike this paper, Dupuy (2015) does not show the equilibrium effect of such shocks, providing only a first order result.⁹

My model extends Roy (1951) in a different direction than the strand of ‘Roy-like’ assignment models (Sattinger, 1975; Teulings, 1995, 2005), in which comparative advantage drives the matching of workers to tasks within a *single* sector and skills are one-dimensional. In this paper, comparative advantage drives between-sectors assignment, whereas within-sector matching is determined by the scale of operation effect. There are, further, a number of papers in the trade literature that build on such ‘Roy-like’ assignment models (Costinot and Vogel, 2010; Sampson, 2014). Those models feature multiple matching markets – countries – but, as labour is assumed internationally immobile, there is no sorting between markets, which is the main focus here. The comparative statics results in this literature focus mostly on the impact of trade, but Costinot and Vogel (2010) consider also a multiplicative shock in the foreign country and find that this increases inequality in both countries. Hence, although the transmission channel is different, the inequality effect of a multiplicative shock is similar to that in my model.

⁸Generally, the type of model considered in Dupuy differs from mine – as firms are one-dimensional, the size of any sector can never be exogenously fixed. However, as Dupuy further assumes that the masses of agents and firms are equal, his specification is equivalent to the special case of my model for which the measure of firms is equal to 1 in each sector.

⁹The equilibrium adjustment process can be thought of as a following chain of events: the shock changes wages, which impacts sorting, which impacts wages, which impacts sorting etc. until a new equilibrium is reached. The results in Dupuy (2015) consider only the first change in sorting, not the subsequent ones. Proposition 6 in this paper does derive the full equilibrium effect for the type of shock considered by Dupuy.

2 Model

There are two populations: agents and firms. There is a measure 1 of agents, each endowed with a skill vector (x_1, x_2) . The distribution function F of skills has compact support $X = [x_1^l, x_1^h] \times [x_2^l, x_2^h]$, is twice continuously differentiable and has strictly positive density on its support. Firms are divided into two sectors and each firm in sector $i \in \{1, 2\}$ is endowed with productivity z_i . The distribution function H_{Z_i} of sector i productivity has compact support $Y_i = [z_i^l, z_i^h]$, is strictly increasing and continuously differentiable. A match between an agent with skill vector (x_1, x_2) and a firm of type z_1 in sector one produces a surplus of $\Pi^1(x_1, z_1)$ and a match between the agent and a firm of type z_2 in sector two produces a surplus of $\Pi^2(x_2, z_2)$, where $\Pi^1 : [x_1^l, x_1^h] \times [z_1^l, z_1^h] \rightarrow \mathbf{R}^+$, $\Pi^2 : [x_2^l, x_2^h] \times [z_2^l, z_2^h] \rightarrow \mathbf{R}^+$ are twice continuously differentiable, strictly increasing in skill, weakly increasing in productivity and weakly supermodular. Note that only the x_1 -coordinate of workers' skill matters in sector 1 and only the x_2 -coordinate matters in sector 2. The size of sector i is normalised to $R^i > 0$. Any agent and firm are free not to match anyone, in which case they receive a reservation utility/profit normalised to 0.

2.1 Copula Formulation

Although so far agent type was represented by a skill vector (x_1, x_2) , it is at least as natural (and will be more convenient) to represent agents by their skill *ranks*, so that agent (u, v) is the $1 - u$ most *talented* in the first dimension and agent v is the $1 - v$ most talented in the second dimension. The rank representation results in a decomposition of the skill distribution into the copula, which fully captures skill interdependence, and marginals – this will play an important role in my comparative statics exercises (Sections 3 to 5).

Denote the marginals of $F(x_1, x_2)$ as $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ and define the *talents* $U = F_{X_1}(X_1)$ and $V = F_{X_2}(X_2)$.¹⁰ The joint distribution of ranks within the population of agents is given by the copula:

$$C(u, v) = F(F_{X_1}^{-1}(u), F_{X_2}^{-1}(v)).$$

$C(\bullet)$ is continuously differentiable and $C_{uv}(\bullet)$ exists, is continuous and strictly positive.

¹⁰In the introduction I used *skill* and *talent* interchangeably, because – there – skill levels were equal to skill ranks. In general, however, skill will refer to skill levels and talent to its rank.

Firms can also be represented by their rank h within sector i . Define $H^i = H_{Z_i}^{-1}(Z_i)$ for $i = 1, 2$. Clearly, H_i is standard uniform distributed. Whenever an agent (u, v) is matched with a firm (h, i) , they produce a surplus of:

$$\pi^1(u, h) = \Pi^1(F_{X_1}^{-1}(u), H_{Z_1}^{-1}(h)),$$

$$\pi^2(v, h) = \Pi^2(F_{X_2}^{-1}(v), H_{Z_2}^{-1}(h)),$$

for $i = 1$ and $i = 2$, respectively. We can easily see that $\pi^1(\bullet)$ and $\pi^2(\bullet)$ are weakly increasing in h , strictly increasing in talent and weakly supermodular.

2.2 (Stable) Matchings and Assignments

I use the stable matching as the equilibrium concept.¹¹ This section defines (stable) matchings, assignments and payoffs.

Definition 1. A *matching* consists of a subset of matched agents $A_A \subset [0, 1] \times [0, 1]$, a subset of matched firms $A_F \subset [0, 1] \times \{1, 2\}$ and a matching function, $\zeta : A_A \rightarrow A_F$.

A matching specifies the types of agents and firms that become matched as well as who they become matched with.

For any measurable subset of firms $B \subset [0, 1] \times \{1, 2\}$ denote its partition into the set of sector one firms and the set of sector two firms as $B^1 = \{(h, 1) \in B\}$ and $B^2 = \{(h, 2) \in B\}$. Recall that the mass of the agents' population is one and the size of sector i is R^i . Hence, the measure of B is $\nu(B) = \int_{B^1} R^1 dh + \int_{B^2} R^2 dh$ and the measure of any measurable subset of agents $E \in [0, 1]^2$ is $\mu(E) = \int \int_E C_{uv}(u, v) du dv$.

Definition 2. A matching $(A_A, A_F, \zeta(\bullet))$ is *measure consistent* if, for any measurable $B \subset A_F$ and its preimage $\zeta^{-1}(B) \subset A_A$, $\nu(B) = \mu(\zeta^{-1}(B))$.

The measure consistency property requires that any subset of matched firms is of the same size as the subset of agents they are matched with (e.g. Legros and Newman, 2002, p. 929), which assures that a matching is 'one-to-one'.¹² Define a *payoff scheme* as a pair of mappings: $w : [0, 1]^2 \rightarrow \mathbf{R}^+$ and $r : [0, 1] \times \{1, 2\} \rightarrow \mathbf{R}^+$.

¹¹Note that for frictionless matching models the concept of a stable matchings is effectively equivalent to the core property (Shapley and Shubik, 1971; Gretskey, Ostroy, and Zame, 1992).

¹²Although matchings are 'one-to-one' here, this model can be reinterpreted in a way that allows for multi-job companies. Specifically, all results would be the same if both sectors consisted of two and more identical, multi-job, hierarchical companies (as in Costrell and Loury, 2004) rather than of infinitely many one-job firms.

The first mapping – w – will be interpreted as the wage function and the second – r – as the profit function.

Definition 3. Given a matching $(A_A, A_F, \zeta(\bullet))$, any associated payoff scheme (w, r) is *feasible* if:

$$\begin{aligned} & \text{for all } (u, v, h, i), \text{ such that } \zeta(u, v) = (h, i) : \quad w(u, v) + r(h, i) \leq \pi^i(u, v, h) \\ & \text{for all } (u, v) \notin A_A \quad w(u, v) = 0 \\ & \text{for all } (h, i) \notin A_F \quad r(h, i) = 0. \end{aligned}$$

Hence, a payoff scheme is feasible if any unmatched firm and agent produce zero and the sum of the firm's profit and the worker's wage in any match does not exceed the surplus they produce.

Definition 4. A matching $(A_A, A_F, \zeta(\bullet))$ is *stable* if it is measure consistent and there exists a payoff scheme that is feasible given $(A_A, A_F, \zeta(\bullet))$, such that for any (u, v, h, i) :

$$w(u, v) + r(h, i) \geq \pi^i(u, v, h). \quad (1)$$

A matching is stable if it is measure consistent and there exists no agent-firm pair that would prefer to be assigned with each other rather than with their current matches.

Definition 5. Any matching $(A_A, A_F, \zeta(\bullet))$ results in an *assignment*, given by the set A_A and an assignment function $\theta : A_A \rightarrow \{1, 2\}$:

$$\theta(u, v) = \begin{cases} 1 & \text{if } \zeta(u, v) \in A_F^1 \\ 2 & \text{if } \zeta(u, v) \in A_F^2. \end{cases}$$

Most of the time the focus will be on the assignment of workers to sectors, which is easily inferred from a matching. An assignment $(A_A, \theta(\bullet))$ is *stable* if there exists a stable matching that results in $(A_A, \theta(\bullet))$.

2.3 Characterisation Strategy

To characterise the stable assignment I deploy a two-step strategy. In the first step, I treat assignment as given and derive the within-sector stable matchings and associated wage functions. This is very similar to the problem first solved by Sattinger (1979). In the second step, I use those wages to find the stable assignment in a manner somewhat similar to Roy's model.

2.3.1 First Step

In this part, I treat the assignment as given and suppress it from notation. Denote the distribution of U conditional on assignment to sector one as $G^1(\cdot)$ and the distribution of V conditional on assignment to sector two as $G^2(\cdot)$. In other words, $G^i(\cdot)$ is the marginal distribution of sector i talent among the workers who joined that sector. Let the *critical abilities* u^c and v^c be the greatest lower bounds of $G^1(\cdot)$ and $G^2(\cdot)$ supports, respectively, and denote the mass of agents in sector i as M^i . Note that $M^i \leq R^i$ by measure consistency. Define the within-sector matching functions as $\zeta^1(u) : [u^c, 1] \rightarrow [0, 1]$, $\zeta^2(v) : [v^c, 1] \rightarrow [0, 1]$. In particular, we can define the positive and assortative matching functions.

Definition 6. The sectoral *positive and assortative matching functions (PAM)* are given by: $P^1(u) = \frac{1}{R^1}(R^1 + M^1(G^1(u) - 1))$ and $P^2(v) = \frac{1}{R^2}(R^2 + M^2(G^2(v) - 1))$.

Proposition 1. Given an assignment $(A_A, \theta(\bullet))$, there exists a stable within-sector matching and the corresponding wage functions must take the form:

$$w^1(u) = \int_{u^c}^u \pi_u^1(r, P^1(r))dr + C^1, \quad (2)$$

$$w^2(v) = \int_{v^c}^v \pi_v^2(r, P^2(r))dr + C^2, \quad (3)$$

where $C^1 \in [0, \pi^1(u^c, \frac{R^1 - M^1}{R^1})]$ and $C^2 \in [0, \pi^2(v^c, \frac{R^2 - M^2}{R^2})]$.

Proposition 1 is essentially a restatement of Sattinger's famous result for strictly supermodular surpluses.¹³ With weak supermodularity more matchings might be stable than just PAM, but they all result in wages of the same form.¹⁴ This is the case as for the set of firms and workers for which $\pi_{uh}^1(\bullet) = 0$ the marginal surplus of each agent's talent in any matching is identical to that holding under PAM.

Thus, for the purpose of finding the stable assignment, we can proceed as if matchings in both sectors were positive and assortative, even if they are not. Clearly, if there are any unmatched agents in a stable matching, competition pushes wages of the worst agents to zero – and contrary, if there are unmatched firms in any of the sectors, profits of the least productive firms are pushed to zero.

¹³Legros and Newman (2002) call 'famous' the result that under weakly supermodular surpluses any stable matching can be supported only by the payoff schemes that support PAM. However, they don't provide any references and their Proposition 3 holds only for one-sided matching markets. Sattinger (1979) shows that the cross-derivative of the surplus function needs to be positive, but his argument holds only for strictly positive cross-derivatives.

¹⁴Definition 2 implicitly allows only for pure matchings, which greatly simplifies notation. This is without loss of generality precisely because all matchings, even impure ones, will still result in wage functions of the form given above.

Lemma 1. In any stable matching it has to be the case that: (i) if $R^1 > M^1$ then $w^1(u^c) = \pi^1(u^c, \frac{R^1 - M^1}{R^1})$ and if $R^2 > M^2$ then $w^2(v^c) = \pi^2(v^c, \frac{R^2 - M^2}{R^2})$; (ii) if $1 - M^1 - M^2 > 0$ then $w^1(u^c) = w^2(v^c) = 0$.

2.3.2 Second Step

Inequality (1) in the definition of a stable matching has to hold for any worker-firm pair, so also for within-sector pairs. Therefore, in any stable matching wage functions need to be of the form derived above. In this part, I use this to find the assignments that result from stable matchings – the stable assignments.

Note that Proposition 1 and positive surplus functions imply that we can have either unmatched agents or unmatched firms, but never both: otherwise there would be a positive measure of both firms and agents earning zero, which contradicts stability. If there are more firms than agents ($R^1 + R^2 > 1$) some firms end up unmatched, because of measure consistency – and thus all agents find a match. Similarly, if firms are scarce ($R^1 + R^2 < 1$) then all find a match, whereas some agents do not. Finally, if $R^1 + R^2 = 1$, then both all firms and all agents will be matched.¹⁵ These facts combined imply that

$$M^1 + M^2 = \min\{R^1 + R^2, 1\}. \quad (4)$$

It is possible that all agents work in a single sector. For this to happen it is necessary that $R^i \geq 1$. Lemma 1 implies that additionally we need $\pi^i(0, \frac{R^i - 1}{R^i}) \geq \pi^j(1, 1)$, as otherwise some agents would strictly prefer to work in sector j . As it turns out, these two conditions are not only necessary, but also sufficient.

Proposition 2. In any stable assignment we have that $M^i = 1$ if and only if $R^i \geq 1$ and $[\pi^i(0, \frac{R^i - 1}{R^i}) \geq \pi^j(1, 1)]$.

Thus, if $R^i \geq 1$ and $\pi^i(0, \frac{R^i - 1}{R^i}) \geq \pi^j(1, 1)$ the unique stable assignment will have all agents working in one sector. Therefore, we need only to characterise stable assignments for cases where this is not true for any sector.

Assumption 1. Both for $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$ it is the case that $R^i < 1$ or $\pi^i(0, \frac{R^i - 1}{R^i}) < \pi^j(1, 1)$.

Assumption 1 together with measure consistency ensure that $M^i \in (0, R^i]$ – so there is a positive measure of agents in each sector that does not exceed

¹⁵Suppose not. Then measure consistency implies that both a positive measure of agents and firms would need to be unmatched, which is impossible.

the measure of firms. The remainder of this paper will focus on cases where Assumption 1 is met.

By definitions of u^c and v^c the mass of matched agents with $(u, v) < (u^c, v^c)$ has to be zero. At the same time, any agent with $u > u^c$ or $v > v^c$ receives a strictly positive wage in one of the sectors; therefore all agents with $(u, v) < (u^c, v^c)$ and only such agents remain unmatched. Thus:

$$C(u^c, v^c) = \max\{1 - R^1 - R^2, 0\}.$$

Moreover, it has to be the case that agents with talent (u^c, v^c) need to earn the same wages in both sectors, as otherwise some agents would want to relocate.

Lemma 2. If Assumption 1 holds, then $w^1(u^c) = w^2(v^c)$.

Clearly, if one of the sectors offers higher wages in the top end of the distribution, it is able to attract all agents who are highly talented in both dimensions. Define the *star abilities* u^* and v^* :

$$\begin{aligned} u^* = 1 \quad \text{and} \quad w^1(1) = w^2(v^*), \quad \text{if } w^1(1) \leq w^2(1) \\ v^* = 1 \quad \text{and} \quad w^1(u^*) = w^2(1), \quad \text{if } w^1(1) > w^2(1). \end{aligned}$$

Hence, any agent with $u \in (u^*, 1]$ strictly prefers to join sector one and any agent with $v^* \in (v^*, 1]$ strictly prefers to join sector two. Note also that it is always the case that $\max\{u^*, v^*\} = 1$.

Define the set $\Gamma = [u^c, u^*] \times [v^c, v^*]$. As $M^1, M^2 > 0$, Γ is non-empty.¹⁶ Agents with $(u, v) \in \Gamma$ will join sector one if $w^1(u) > w^2(v)$ and sector two if $w^2(v) < w^1(u)$. Define a function $\psi : [v^c, v^*] \rightarrow [u^c, u^*]$ such that:

$$w^1(\psi(v)) = w^2(v). \tag{5}$$

Any agent in Γ will strictly prefer sector one if $u > \psi(v)$ and sector two if $u < \psi(v)$.

Remark 1. The quintuple $(u^c, v^c, u^*, v^*, \psi(\bullet))$ fully characterises the stable assignment. Agents with $(u, v) < (u^c, v^c)$ are unmatched. Sector 1 is populated by agents with: (i) $v < v^c$ and $u > u^c$; (ii) $(u, v) \in \Gamma$ and $\psi(v) < u$; (iii) $u > u^*$. Sector 2 is populated by agents with: (i) $v > v^c$ and $u < u^c$; (ii) $(u, v) \in \Gamma$ and $\psi(v) > u$; (iii) $v > v^*$. The set of remaining agents is of zero measure.

¹⁶ As $M^i > 0$, it has to be the case that $v^c, u^c < 1$. Wage functions are strictly increasing and therefore $w^2(1) > w^1(u^c)$ and $w^1(1) > w^2(v^c)$, which implies that $v^* > v^c$ and $u^* > u^c$.

To solve for this quintuple, I first derive within-sector talent distributions.

Lemma 3. Given $(u^c, v^c, u^*, v^*, \psi(\bullet))$, the marginal distributions of talent in sectors one and two are, respectively:

$$G^1(u) = \begin{cases} 0, & u < u^c, \\ \frac{\int_{u^c}^u C_u(r, \psi^{-1}(r)) dr}{M^1}, & u \in [u^c, u^*], \\ G^1(u^*) + \frac{u-u^*}{M^1}, & u \in (u^*, 1]; \end{cases} \quad G^2(v) = \begin{cases} 0, & v < v^c, \\ \frac{\int_{v^c}^v C_v(\psi(r), r) dr}{M^2}, & v \in [v^c, v^*], \\ G^2(v^*) + \frac{v-v^*}{M^2}, & v \in (v^*, 1]. \end{cases}$$

Substituting the marginal distributions back into wage functions and combining all the results so far, we arrive at the following set of equations and conditions:

$$\text{for } v \in [v^c, v^*]: \quad \int_{u^c}^{\psi(v)} \pi_u^1\left(t, \frac{R^1 - M^1 + \int_{u^c}^t C_u(r, \psi^{-1}(r)) dr}{R^1}\right) dt \quad (6)$$

$$= \int_{v^c}^v \pi_v^2\left(t, \frac{R^2 - M^2 + \int_{v^c}^t C_v(\psi(r), r) dr}{R^2}\right) dt, \quad (7)$$

$$C(u^c, v^c) = \max\{1 - R^1 - R^2, 0\}, \quad (7)$$

$$\max\{u^*, v^*\} = 1, \quad (8)$$

$$\int_{u^c}^{u^*} C_u(r, \psi^{-1}(r)) dr + 1 - u^* = M^1, \quad (9)$$

$$\int_{v^c}^{v^*} C_v(\psi(r), r) dr + 1 - v^* = M^2, \quad (10)$$

$$M^1 + M^2 = \min\{R^1 + R^2, 1\}, \quad (11)$$

$$M^1 \in (0, R^1] \text{ and } M^2 \in (0, R^2], \quad (12)$$

$$M^1 < R^1 \Rightarrow \pi^1\left(u^c, \frac{R^1 - M^1}{R^1}\right) \leq \pi^2\left(v^c, \frac{R^2 - M^2}{R^2}\right), \quad (13)$$

$$M^2 < R^2 \Rightarrow \pi^2\left(v^c, \frac{R^2 - M^2}{R^2}\right) \leq \pi^1\left(u^c, \frac{R^1 - M^1}{R^1}\right). \quad (14)$$

Equations (9) and (10) result from the fact that $G^1(1) = G^2(1) = 1$. Conditions (13) and (14) follow from Lemmas 1, 2 and Proposition 1.

2.3.3 Existence and Uniqueness

A solution to the set (6)-(14) gives us $(u^c, v^c, u^*, v^*, \psi(\bullet))$, as well as M^1 and M^2 , and thus fully characterises a stable assignment. This characterisation of stable assignments will lay at the heart of the subsequent comparative statics analysis. Note that each of those equations and conditions need to hold for any stable assignment and thus any stable assignment has to be represented by a solution to

(6)-(14).¹⁷

Theorem 1. If Assumption 1 holds, then solution to the set (6)-(14) exists, is unique and fully characterises the unique stable assignment as specified in Remark 1. If Assumption 1 does not hold, then in the unique stable assignment all agents work in the sector for which $\pi^i(0, \frac{R^i-i}{R^i}) \geq \pi^j(1, 1)$.

The proof relies on constructing a map, the fixed point of which is equivalent to the solution of (6) and finding a norm for which this map is a *contraction mapping*.¹⁸ This proves that $\psi(\cdot)$ is unique *given* (u^c, v^c, M^1, M^2) – and also continuous in those variables. Then showing existence and uniqueness is merely a matter of proving that the remaining equations have a unique solution given the function $\psi(u^c, v^c, M^1, M^2)$. The existence of a stable matching – and thus a stable assignment – could be alternatively shown by rewriting the model as a special case of Gretsky et al. (1992). The uniqueness result is, however, new – it does not follow from the existing uniqueness results for stable matchings (e.g. Chiappori, McCann, and Nesheim, 2010), as it is possible that there are multiple stable matchings in this model – all resulting in the same assignment.

Trivially, the existence of a stable assignment implies the existence of stable matchings.

2.3.4 Sattinger and Roy

The first step in my characterisation strategy is very similar to Sattinger (1979), the second to Roy (1951). This is not a coincidence: in fact, both one-sector matching and Roy-like models are nested within this framework.¹⁹ In the case of Sattinger’s model, that’s fairly obvious: if one of the sectors is sufficiently more productive than the other and there is an abundance of its firms (so if Assumption 1 does not hold), then all agents will work in that industry and the model collapses to just one sector.

As for Roy-like models, suppose that firms in both sectors are identical, in

¹⁷This follows from the arguments in Sections 2.3.1 and 2.3.2, in particular Proposition 1, Lemmas 1 and 3 and the discussions of Equation 4 and Assumption 1.

¹⁸The norm I use is Bielecki’s norm for a high-enough parameter λ .

¹⁹The actual models by Sattinger and Roy are not, strictly speaking. In the former case, the reason is that Sattinger allows for cases in which both firms and agents are unemployed, which is ruled out here by the assumption of positive surpluses – so only certain special cases of his model are nested. In the latter, the reason is that Roy uses bi-variate log-normal distribution of skills, which is not defined over an rectangle – however, we could get an arbitrarily good approximation of Roy’s model, by using bi-variate log-normal distribution, truncated arbitrarily high and arbitrarily close to zero.

which case the surplus produced by any match depends on the agent's skill only.²⁰ If, on top of that, there is an abundance of firms in each sector, then firms have no market power. Hence, the agents receive the entire surplus and their wage does not depend on the assignment, exactly as in Roy's model. In other words, Roy-like models can be seen as two-sector matching models in which all firms from the same sector are homogeneous. In such a case, all within-sector matchings are stable.

3 Changes in Marginal Distribution of Skill

In this section I study the effects of an increase in the *spread* of the marginal distribution of sector one skill, whilst keeping the copula and sector two specification unchanged.

I start by defining the key concepts of the analysis, that is vertical differentiation, talent supply and talent demand. In all comparative statics exercises in this paper I consider two matching problems – the *old* and the *new* one – that meet all conditions from Section 2, including Assumption 1. To distinguish between the new and old problem, I introduce the parameter ρ ; the old problem is denoted by ρ_1 and the new one by ρ_2 . For example, $u^c(\rho_1)$ is the old critical ability in sector one, whereas $v^*(\rho_2)$ is the new star ability in sector two.

Definition 7 ((Bickel and Lehmann, 1979)). A distribution $F_X(x, \rho_2)$ is (*strictly*) *more spread out in Bickel-Lehman sense* than distribution $F_X(x, \rho_1)$ if:

$$F_X^{-1}(u_2, \rho_2) - F_X^{-1}(u_1, \rho_2)(>) \geq F_X^{-1}(u_2, \rho_1) - F_X^{-1}(u_1, \rho_1) \text{ for all } 0 \leq u_1 < u_2 \leq 1.$$

A Bickel-Lehman spread of the marginal skill distribution increases the difference in skill for any two levels of talent. Unfortunately, this notion of spread is not invariant under increasing transformations and, hence, depends on the – largely arbitrary – choice of skill units.²¹ For this reason, I focus directly on the spread of surplus, which does not depend on the units in which skills are measured.

Definition 8. Sector one surplus becomes (*strictly*) *more spread out* if the new distribution of surplus $S = \Pi^1(X_1, H_{z_1}^{-1}(h))$ is (*strictly*) more spread out in the

²⁰This implies $\Pi_z^1(\bullet) = \Pi_z^2(\bullet) = \Pi_{xz}^1(\bullet) = \Pi_{yz}^2(\bullet) = 0$, which is allowed by my assumptions.

²¹To see why this is problematic, take the example of years in education, which is a popular proxy for skill. Assuming, somewhat implausibly, that people are homogeneous in the time spent on learning in any given year of education, hours of learning are an increasing transformation of years in education. However, as people tend to spend more time learning in the later stages of their education, this transformation is not linear, but convex.

Bickel-Lehman sense than the old distribution of S , for firms of all ranks $h \in [0, 1]$.

Any surplus spreading change of the marginal distribution of skill will be called a *real spread* of skill.²² Note that an increase in surplus' spread is equivalent to a rise in *vertical differentiation of workforce*, as it increases the marginal surplus of talent ($\pi_u^1(u, h, \rho_2) \geq (>) \pi_u^1(u, h, \rho_1)$ for all $(u, h) \in [0, 1]^2$).

It is worth noting that an increase in surplus' spread can be driven not only by real spreads of the marginal distribution of skill, but also by changes in the surplus function itself (as in the introduction) and in the distribution of productivity (see Section 4). Unless stated otherwise, the results in this section apply for all surplus' spreading changes of sector one specification.

I define sectoral *supply and demand* for talent of level t in cumulative terms, as, respectively, the mass of agents with talent greater than t who join sector i , and the mass of sector i firms who demand agents with talent greater than t , both for a given wage function $w(\bullet)$. Talent demand (supply) *shifts up* if, keeping wages constant, the cumulative demand (supply) function improves for all possible talent levels. Formal definitions of (cumulative) talent supply and demand (shifts) can be found in Appendix C.

Definition 9. *More talent is supplied* to sector i if, for any talent level t , the equilibrium measure of agents with talent greater than t increases. Formally:

$$M^i(\rho_2)(1 - G^i(t, \rho_2)) \geq M^i(\rho_1)(1 - G^i(t, \rho_1)) \quad \text{for all } t \in [0, 1]. \quad (15)$$

More high talent is supplied to sector i , if there exists a $t^* < 1$ such that Equation (15) holds for all $t \in (t^*, 1]$. This is strict, if (15) holds strictly for some t .

There is more talent supplied in equilibrium if the measure of agents with talent greater than t increases for any t . Note that whenever I write that *talent supply increases* I refer to the equilibrium measure of talent supplied, not to a shift in talent supply.

²²Real spread is a stronger notion than Bickel-Lehman spread, in the sense that it requires a continuum of increasing transformations of the marginal distribution to become more spread out in the Bickel-Lehman sense. However, for the special case of a multiplicative surplus function – $\Pi^1(x, z) = xz$ – real spread is equivalent to Bickel-Lehman spread. And more generally, if there exists such a unit of skill that surplus is linear in it for all $z \in [z_l^1, z_h^1]$, then real spread is equivalent to Bickel-Lehman spread of skill (measured in this unit).

3.1 Scarce Jobs

I will consider the cases of scarce ($R^1 + R^2 \leq 1$) and abundant ($R^1 + R^2 > 1$) jobs separately. The former is much simpler – and possibly more realistic – and thus serves as a good starting point. In particular, scarcity of jobs implies that all firms are matched – which in turn means that sector sizes are fixed and the PAM function in each sector is equal to the distribution of skill.

Corollary 1. If $R^1 + R^2 \leq 1$ then $P^1(u) = G^1(u)$, $P^2(v) = G^2(v)$ and $M^i = R^i$. If jobs are strictly scarce ($R^1 + R^2 < 1$), then additionally $w^1(u^c) = w^2(v^c) = 0$.

The first two claims are implied by Equations (6)-(14), whereas the last follows trivially from Lemma 1. The following Proposition, as well as all comparative statics results in this paper, follows from Theorem 2 in Appendix B, which links changes in the surplus function of the copula formulation with changes in sectoral talent supply.

Proposition 3. If jobs are scarce and sector one surplus becomes more spread out, then (i) the distribution of talent in sector one improves in first order stochastic dominance sense; (ii) the distribution of talent in sector two deteriorates in first order stochastic dominance sense. If, further, the spread is strict, then strictly more talent is supplied to sector one and strictly less to sector two.

With scarce jobs, the measure of matched firms is fixed and the levels of agents' skills do not matter for talent demand and sorting.²³ An increase in surplus' spread makes talented workers relatively underpaid, which shifts up the demand for talent and, ignoring relocation, increases wages. In general equilibrium, this attracts agents from the other sector and translates into higher equilibrium supply of talent in sector one (lower in sector two).

To bring the most interesting results into focus, in the discussion on wages I focus on strict increases in the spread of surplus; all results extend easily to the more general case. It is also worth noting that with strict supermodularity results (i) and (iii) below hold strictly.

Proposition 4. If jobs are scarce and sector one surplus becomes strictly more spread out, then (i) in sector two, wages increase for all agents and the higher

²³To see this clearly, consider the following extreme example (it violates the assumption of continuous skill distribution, but nevertheless it conveys very well the logic of what is happening). Suppose that in the old matching problem, every agent had the same sector one skill $x(\rho_1) > x^h(\rho_2)$. Thus, workers were identical for sector one firms', which paid them zero and all talented workers were in sector two. Any strict spread of sector one skills results in positive sector one wages and an improvement in its talent-pool – despite the strict decrease in levels.

the talent the greater the increase; (ii) in sector one, wages increase strictly for a positive mass of the most talented agents; and (iii) in both sectors the range of the wage distribution increases, strictly in sector one. If jobs are strictly scarce, then, further, the increases in wages and wage range in sector two are strict and sector one wages fall strictly for a positive mass of the least talented agents.

As cumulatively less talent is supplied to sector two, there are fewer highly talented agents in that industry, regardless of how ‘high talent’ is defined. This raises the relative price of talent and the spread of wage increases in the Bickel-Lehman sense.²⁴ As the lowest wage is fixed at the reservation level, this implies that wages increase for all talent levels. In sector one, the rise in wage inequality happens in a weaker sense (range increases), as the increased differentiation raises the relative price of talent, but the greater supply of talent works in the opposite direction. For agents with highest talent, increased differentiation dominates and their wages rise.²⁵ And, with strictly scarce jobs, the increase in talent supply dominates for the least talented agents, who are pushed down the ladder so much that their wages fall.²⁶

In the case of sector two we can say a little bit more about measures of inequality other than range. Keeping talent distribution constant, the increase in the spread of sector two wage raises its variance.²⁷ This means that variance increases for the *incumbents*, i.e. the agents who worked in sector two both before and after the shock.²⁸ It does not, however, necessarily imply an overall increase in sector two variance, as the distribution of talent does change.²⁹ Keeping wage function constant, the fall in talent supply has an ambiguous effect on variance. Hence, the overall effect of an increase in sector one vertical differentiation sector two wage variance is ambiguous.

Unlike wages, profits depend not only on talent supply and demand, but also on how much firms can produce and how dissimilar they are. These are determined not so much by the spread of skills, as by their levels: the former trivially and the latter as, because of supermodularity, the more skilled the agents are, the greater

²⁴Specifically, the new distribution of $W = w^2(X)$ is more spread out than its old distribution, see (i) in the proof of Proposition 4.

²⁵To see this, note that top wages increase in sector two and yet the most talented agents are more likely to join sector one than previously.

²⁶This is not necessarily true for weakly scarce jobs, as then it is possible that the critical ability is unchanged (and equal to zero), which implies that, in her case, the two opposing forces balance each other out.

²⁷Follows from 3.B.25 in Shaked and Shanthikumar (2007).

²⁸Formally, the set of incumbents is defined as $I = \{(u, v) : \theta(u, v, \rho_1) = 2 \text{ and } \theta(u, v, \rho_2) = 2\}$.

²⁹In fact, it is the change in talent distribution that spreads out sector two wage in the first place.

is the difference in surplus produced by high and low productivity firms and the more dissimilar they become. Therefore, in order to say anything conclusive about sector one profits, we need to know what happens to surplus' levels.

Definition 10. Sector one surplus rises (falls) if, for all $(u, h) \geq (u^c(\rho_1), \frac{R^1 - M^1}{R^1})$, $\pi^1(u, h, \rho_2) \geq (\leq) \pi^1(u, h, \rho_1)$.

Keeping assignment constant, this condition ensures an increase in sector one total output.³⁰ It follows trivially from Definition 8 that a real spread results in a rise in surplus levels iff $F_X^{-1}(u^c(\rho_1), \rho_2) \geq (\leq) F_X^{-1}(u^c(\rho_1), \rho_1)$.

Proposition 5. If jobs are scarce and sector one surplus becomes strictly more spread out, then output and profits fall in sector two. If the change in surplus is caused only by the marginal distribution of skill and, further, surplus rises, then profits grow for all sector one firms, and output produced in sector one increases, as well as total output produced in the economy.

In sector two, we have only the general equilibrium effect of decreased talent supply, which depends on sector one skills' spread, not levels. In sector one, firms become more dissimilar due to the rise in surplus' levels, the spread of profits increases and, as the least productive firms produce higher surplus and are better off, profits increase. Relocation increases talent supply and enhances this effect.

Total output produced in sector one increases with certainty only if both surplus' spread and levels increase; otherwise the impact on sector one total output is ambiguous. In particular, a change that makes sector one less productive but more differentiated can increase the output of that sector (recall the example from the introduction).

3.2 Abundant Jobs

If jobs are abundant ($R^1 + R^2 > 1$), skill levels do play a role, as they influence the demand for talent at the extensive margin. If agents become less skilled in the X_1 dimension, sector one firms become less competitive and might leave the market, which – on its own – shifts talent demand down. As the increase in spread shifts the demand for talent up, the final effect on relocation is ambiguous.

To address this, I focus on changes that increase both the spread and levels of surplus. This includes, for example, multiplicative shocks. It follows from

³⁰Sector one output is given by $\int_{u^c}^1 \pi^1(r, P^1(r)) dP^1(r)$. Keeping assignment constant $P^1(\cdot)$ and u^c remain unchanged and the statement follows trivially.

Theorem 2 in Appendix B that they result in an increase in talent supply in sector one and a fall in sector two.

Proposition 6. A simultaneous increase in sector one surplus' levels and spread results in more talent being supplied to sector one and less to sector two. If the spread is strict, then the changes in talent supply are strict as well.

Simultaneous increases in real spread and skill levels have, ignoring relocation, a positive effect on sector one wages, as they shift the demand for talent up.³¹ Thus, in general equilibrium, more talent is attracted to sector one, at the expense of sector two.

Note also that it is still the case that if an increase in the real spread of X_1 is strong enough, it can increase the supply of high talent in sector one even if X_1 falls for all talent levels.³² Finally, it follows from the definition of the supply of talent that employment raises in sector one and falls in sector two.

With abundant jobs, demands shifts also at the extensive margin; furthermore, in the important special case of Roy-like models the intensive margin does not matter for the equilibrium supply of talent at all.³³ In general, shifts of demand at the intensive margin increase the relative market power of talented workers, allowing them to receive a greater share of surplus. However, in Roy-like models, firms are not sufficiently heterogeneous to have any market power at all, as each firm can always be replaced by an identical, unmatched company. Therefore, workers always receive the entire surplus, regardless of how differentiated they are. More generally, vertical differentiation of workers has an impact on the stable assignment only if firms heterogeneity is substantial enough to allow at least some of them a degree of market power.

Proposition 7. If jobs are abundant and surplus both increases and becomes strictly more spread out, then all sector two wages improve, as well as wages of the most talented sector one agents.

In sector two, less talent is supplied and thus, again, the relative price of talent

³¹See Proposition 10 in Appendix C.

³²To see this, recall the example from footnote 23 and assume further that the new matching problem is symmetric and that $R^1 > 1$. Then, as long as the old, common skill level is such that $\Pi^1(x(\rho_1), H_{Z^1}^{-1}(\frac{R^1-1}{R^1})) < \Pi^2(y_h, z_h^2)$, we have that there will be a positive mass of agents working in sector two even before the change. As all agents used to be identical for sector one firms, they offered the same wage and all sector two agents were earning more than that. Thus, $v^c(\rho_1) = v^*(\rho_1) < 1 = v^*(\rho_2)$, which means that the most talented agents join sector one.

³³A formal result is provided in the Online Appendix.

increases and wages rise.³⁴ In sector one, for the most talented workers, the shift in talent demand dominates the increase in talent supply. However, the end effect is ambiguous for the least talented workers and, hence, it is possible that they are worse off despite the improvement in surplus' levels.

The impact on wage range is ambiguous in both sectors, as the increase in surplus levels may, in certain cases, be equality-enhancing. For details, see Online Appendix, where I derive formally the results for isolated increases in surplus' spread and surplus' levels. In particular, I show that if there is no shift in talent demand at the extensive margin, then the effects of an increase in talent differentiation are qualitatively very similar to the scarce jobs case.

Proposition 8. If jobs are abundant and surplus both increases and becomes strictly more spread out, then profits increase for all sector one firms and decrease for all sector two firms. Sector one as well as the whole economy produce more output, but sector two produces less.

Firms' gains from the increase in surplus are always at least as big as their losses from increased demand – and even the lowest profits increase.³⁵ Propositions 5 and 8 imply jointly that simultaneous increases in sector one surplus and differentiation always result in an increase in sector one profits and a decrease in sector two profits, irrespective of whether jobs are scarce or abundant.

4 Changes in the Distribution of Productivity

In this section, I study the effects of improvements in the distribution of firms in the first order stochastic dominance sense. The most natural interpretation of such changes is the introduction of a more efficient technology, although trade liberalisation can have a similar effect (see Melitz, 2003; Sampson, 2014).³⁶ First order stochastic dominance is a well-known notion; in order to sharpen some of the following results I additionally define strict first order stochastic dominance.³⁷

³⁴It is worth noting, that although the increase in relative price of talent spreads wages out in the Bickel-Lehman sense, it does not necessarily result in a widening of the pay gap between top and bottom workers. The reason is that critical ability in sector two may rise – as the measure of active sector two firms falls.

³⁵Differently than in the scarce jobs case, even if skill levels are strictly higher it is possible that sector one profits increase weakly – precisely because of the shift in demand.

³⁶Note, however, that in Melitz model trade liberalisation changes also the mass of firms, which is not the case here.

³⁷Strict first order stochastic dominance requires a strict increase in both the upper lower and the lower upper bounds of the productivity distribution.

Definition 11. A distribution $H_{Z^1}(z, \rho_2)$ *strictly first order stochastically dominates* distribution $H_{Z^2}(z, \rho_1)$ iff $H_{Z^1}^{-1}(h, \rho_2) > H_{Z^1}^{-1}(h, \rho_1)$ for all $0 \leq h \leq 1$.

Crucially, keeping the surplus function and skill distribution constant, an improvement in firms' productivity implies an increase in not only surplus' levels, but also spread – as $\Pi^1(\bullet)$ is supermodular, the difference in surplus produced by agents of any skill levels increases in the productivity z . Note that the resulting increase in spread can be strict only if the surplus function is strictly supermodular, which is assumed in the remainder of this section.

Corollary 2. If sector one productivity distribution (strictly) improves in first order stochastic dominance sense, then surplus both increases and becomes (strictly) more spread out.

In other words, an improvement in productivity distribution not only makes a sector more productive, but also more differentiated. This shifts the demand for talent up. Therefore, all Section 3 talent supply and wages results hold also for technological improvements. Consequently, if jobs are strictly scarce, adoption of more efficient technologies makes the least talented workers in that sector worse off (Proposition 4). This makes sense intuitively: under supermodularity, more talented agents can use better technologies more efficiently. It also implies that technological change which is restricted to just one sector can increase wage inequality in many industries. This suggests that, for example, Rosen's (1981) superstar effect could be driving the increases in wage inequality also in sectors that are not directly affected by the improvements in communication technology.

Improvements in productivity distribution have an unambiguous effects on total output and sector two profits. In the latter case, the fall in talent supply decreases profits (Proposition 5). Total output produced in the economy grows, for the usual reasons: fixing the assignment, total output increases; and moving to the new stable assignment improves it further. Sector two output falls, due to lower supply of talent and, hence, sector one output has to increase.

The impact of a technological improvement on sector one profits is ambiguous, as it depends also on whether firms become more or less similar. If productivity simultaneously improves and becomes less spread out, firms produce more, but the bargaining power of the more productive ones deteriorates. In the extreme case of abundant jobs and identical firms, profits are zero: thus, even if the change from the original distribution to the common productivity constitutes an improvement, it may nevertheless result in lower profits for all sector one firms.

5 Changes in Interdependence

In this section, I study the effects of an increase in skill interdependence, whilst keeping skill marginals and firm productivity constant. To make this problem tractable, I restrict attention to matching problems with symmetric copula formulations.

Definition 12. The copula formulation of a matching problem is *symmetric* iff: (i) $C(u, v) = C(v, u)$ for all $(u, v) \in [0, 1]^2$; (ii) $\pi^1(u, h) = \pi^2(u, h)$ for all $(u, h) \in [0, 1]^2$ and (iii) $R^1 = R^2$.

Note that a symmetric original formulation implies that the copula formulation is symmetric as well. However, matching problems with asymmetric original formulations can have symmetric copula formulations.³⁸ Symmetry implies trivially that $u^c = v^c$, $u^* = v^*$ and $\psi(v) = u$ characterises a stable assignment; and as by Theorem 1 the stable assignment is unique, Corollary 3 follows.

Corollary 3. If the copula formulation of a matching problem is symmetric then (i) $v^c = u^c$ is given by the solution to $C(v^c, v^c) = \max\{1 - 2R^1, 0\}$; (ii) $v^* = u^* = 1$ and (iii) $\psi(v) = u$ for $v \in [v^c, v^*]$. This fully characterises the unique stable assignment, as described in Remark 1.

In the symmetric case, agents simply choose the sector which uses the skill in which they are more talented. Clearly, this results in identical talent (but not necessarily skill) distributions, wages and sector sizes in both industries.

Definition 13 (Scarsini, 1984). The joint distribution $F(\bullet, \rho_2)$ is *more concordant* than $F(\bullet, \rho_1)$ if: $C(u, v, \rho_2) \geq C(u, v, \rho_1)$ for all $(u, v) \in [0, 1]^2$.

The concordance ordering formalises the idea of greater interdependence, as higher concordance implies that large values of X_1 are more likely to go with large values of X_2 . Note that an increase in concordance is equivalent to a fall in overall supply of talent, as there are fewer workers who are highly talented in at least one dimension.³⁹

³⁸For example, consider the following original formulation: the distribution of skill is $F(x, y) = \frac{1}{2}(y - 10)x^3$ with support on $[0, 1] \times [10, 12]$; the firms' productivity is standard uniform distributed in both sectors; the surplus functions are $\Pi^1(x, z^1) = x^3(1 + z^1)$ for $(x, z^1) \in [0, 1]^2$ and $\Pi^2(y, z^2) = \frac{1}{2}(y - 10)(1 + z^2)$ for $(y, z^2) \in [10, 12] \times [0, 1]$ and sizes of sectors are $R^1 = R^2 > 0$. This is asymmetric. It is easy to verify that $F_X(x) = x^3$ and $F_Y(y) = \frac{y-10}{2}$. Thus, the corresponding copula formulation is described by: $C(u, v) = uv$, $\pi^1(u, h) = uh + u$, $\pi^2(v, h) = vh + v$ and $R^1 = R^2 > 0$, which is symmetric.

³⁹It follows immediately from the definition of concordance that the measure of agents with talent lower than (u, v) goes up for any (u, v) .

Proposition 9. Consider a matching problem with a symmetric copula formulation and suppose that the copula becomes more concordant. Then, in each sector: (i) the distribution of talent deteriorates in first order stochastic dominance sense; (ii) wages increase for any talent; (iii) wage range decreases; (iv) profits fall for all firms and (v) the output produced in each sector decreases.

A more concordant skill distribution implies a downward shift in talent supply in both sectors, which directly translates into less talent being supplied in equilibrium. This decreases profits and output, but increases wages. The lowest wage, however, remains unchanged and thus wage range goes up.

It is easy to verify that for asymmetric copula formulations a change in concordance does not imply a shift in talent supply in both sectors. Hence, in general, an increase in interdependence does not necessarily reduce the supply of talent in both sectors (in the sense of Definition 9). However, as it does result in a fall in the overall supply of talent, total output produced in the economy falls also in the asymmetric case.

6 Conclusions

In this paper I have developed a tractable two-sector matching model with two skill dimensions, each used exclusively by one of the sectors. My goal was to clarify what determines sectoral supply of talent in a setting with two-sided heterogeneity and more than one skill dimension. The main insight is that sectoral demand for talent – and thus also its equilibrium supply – is determined not only by the absolute productivity in this industry, but also by vertical differentiation of its workforce. Therefore, a sector can expand even if all of its workers become less productive, due to talent supply effects. Moreover, sectoral adjustments in the supply of talent transmit wage inequality across sectors. An increase in top wages in one sector attracts additional talent, which increases its relative price in the other sector. Finally, I have shown that in the symmetric case, an increase in skill interdependence shifts down the supply of talent in both sectors, which naturally leads to less talent being supplied in the equilibrium to both industries.

I have also developed a novel, direct characterisation of stable matchings in multi-sector models. This characterisation, which relies on fixed-point rather than optimal transport theory is the basis of my comparative statics analysis. However, this paper does not exploit all of its advantages. In particular, this approach can also handle settings in which the surplus produced by any individual match

depends on the assignment of agents to sectors – so settings with inter-sector externalities. This property is utilised in Gola (2015), where the externalities result from status concerns.

It is worth noting that the sectors can be also interpreted as countries, which produces further insights. Recall the example from the introduction, think of Germany and the UK and suppose that a negative surplus shock (of the same functional form as in that example) hits the UK. This expands UK’s economy and decreases the variance of British wages, as all of the negative consequences are exported to Germany. Therefore, any policy that results in this shock is beneficial for the UK. If both countries had the option of implementing such a policy and cared only about output, this would result in a game of the following strategic form.⁴⁰

		Germany	
		No shock	Shock
UK	No shock	0%, 0%	-0.54%, 0.15%
	Shock	0.15%, -0.54%	-0.43%, -0.43%

Note that payoffs are expressed in percentage change from the original output level. Clearly, introducing the shock is the dominant strategy for both countries. This implies that international competition for talent can create strategic incentives for countries to increase inequality and decrease output (or, more likely, increase it by less than possible). However, this conclusion needs to be taken with a pinch of salt. First of all, the lack of relocation costs is a very limiting assumption in any migration application. Secondly, the assumption of one-dimensional skill within a matching market makes less sense in the context of countries. Addressing these issues is left for future research.

A Stable Matchings and Assignments

Proof of Proposition 1. I do the proof for sector one, sector two is analogous. First, I will show that PAM is indeed stable with the wage function postulated in the Proposition. Then I will use this and Proposition 3 in Chiappori et al. (2010) to show that wages of the postulated form are indeed the only stable ones.

Suppose $r^1(P^1(u)) = \pi^1(u, P^1(u)) - w^1(u)$. Then the proposed wage scheme is feasible. Note that the PAM function trivially meets the measure consistency

⁴⁰This refers to the following Prisoner’s Dilemma game: the economy is as described in the example and country $i = 1, 2$ chooses between $\Pi^1(\rho_2) = \Pi^1(\rho_1)$ and $\Pi^1(\rho_2) = (x_i + 1 - 0.15(1 - x_i)^4)$. They move simultaneously and each countries’ payoff is its output.

condition.⁴¹ Hence, by Definition 4, PAM with the postulated wage is stable as long as for any worker-firm pair $(u_1, P^1(u_2))$ it is the case that:

$$w^1(u_1) + \pi^1(u_2, P^1(u_2)) - w^1(u_2) \geq \pi^1(u_1, P^1(u_2)).$$

This is met as long as:

$$\int_{u_2}^{u_1} \int_{P^1(u_2)}^{P^1(r)} \pi_{uh}^1(r, s) ds dr \geq 0. \quad (16)$$

As $P^1(\cdot)$ is strictly increasing, if $u_2 \geq u_1$ we have $P^1(u_2) \geq P^1(r)$ for all $r \in [u_1, u_2]$; and if $u_2 < u_1$ we have $P^1(u_2) < P^1(r)$ for all $r \in [u_2, u_1]$. Therefore, condition (16) is met due to supermodularity of $\pi^1(\bullet)$; hence, PAM with $w^1(\cdot)$ is stable.

The one sector matching setup is trivially a special case of the assignment model specified in Chiappori et al. (2010). Label the firms as buyers and the workers as sellers. Then this model meets the conditions of the semi-convex buyer setting from Chiappori et al. (2010).⁴² Hence, their Proposition 3 holds. This implies that for any stable matching the marginal profits are equal to r_h^1 almost everywhere. This in turn implies that the wage function for any stable matching is of the proposed form, as $w^1(u) = \pi^1(u, P^1(u)) - r^1(P^1(u))$. \square

Proof of Lemma 1. (i) Consider sector one and suppose that the firm with $h = \zeta_1(u^c)$ makes a positive profit and thus $r^1(\zeta_1(u^c)) > 0$. But as $R^1 > M^1$ we can always find an unmatched firm with h_2 , such that $\pi^1(u^c, \zeta_1(u^c)) - \pi^1(u^c, h_2) < r^1(\zeta_1(u^c))$. We can then use $\pi^1(u^c, \zeta_1(u^c)) = w^1(u^c) + r^1(\zeta_1(u^c))$ to get $w^1(u^c) - \pi^1(u^c, h_2) = w^1(u^c) + r^1(h_2) - \pi^1(u^c, h_2) < 0$, which contradicts stability. Therefore, $r^1(\zeta_1(u^c)) = 0$, which means that $w^1(u^c) = \pi^1(u^c, \zeta_1(u^c))$. It follows from proof of Proposition 1 that $\pi^1(u^c, \zeta_1(u^c)) \leq \pi^1(u^c, \frac{R^1 - M^1}{R^1})$. Suppose this holds strictly. Then by property (b) of the surplus function, it follows that $\zeta_1(u^c) < \frac{R^1 - M^1}{R^1}$. But measure consistency implies that then there needs to exist some unmatched firm with $h' \geq \frac{R^1 - M^1}{R^1}$, which trivially contradicts stability. Therefore, $\pi^1(u^c, \zeta_1(u^c)) = \pi^1(u^c, \frac{R^1 - M^1}{R^1})$. Sector 2 is analogous.

(ii) $1 - M^1 - M^2 > 0$ implies that there are some unmatched agents who earn

⁴¹Consider any interval $[h_l, h_h] \subseteq A_F^1$. The measure of firms in this interval is $R^1(h_h - h_l)$. By Definition 6 the measure of agents matched with firms in this interval is $R^1 h_h - R^1 + M^1 - R^1 h_l + R^1 - M^1$, which gives $R^1(h_h - h_l)$ as well. Finally, note that any measurable set $B \subseteq A_F^1$ can be partitioned into intervals and sets of measure zero.

⁴²Surplus function is twice differentiable, $[0, 1]$ and $[u^c, 1]$ are smooth manifolds and standard uniform distribution puts zero mass on any $h \in [0, 1]$. See Definition 4 in Chiappori et al. (2010) and its discussion.

0. Suppose that $w^1(u^c) > 0$. But then we can always find some $u \in [0, u^c)$ such that $r^1(\zeta_1(u^c)) < \pi^1(u, \zeta_1(u^c))$ (by continuity of $\pi^1(\bullet)$), which contradicts stability. Therefore $w^1(u^c) = 0$ and the same is true for $w^1(v^c)$. \square

Proof of Proposition 2. I start showing “only if” for sector one and then move to “if” – the proofs for sector two are analogous.

R^1 needs to be weakly greater than 1 from measure consistency. Suppose that $\pi^2(1, 1) > \pi^1(0, \frac{R^1-1}{R^1})$ and $M^1 = 1$. Therefore, for any arbitrarily small $\epsilon > 0$ there has to exist a positive measure of unmatched sector one firms with $h' \in [1 - \epsilon, 1]$. As density is strictly positive over the entire domain and wage and surplus functions are continuous it follows that there has to exist a positive mass of agents such that $w^2(v) + r(h', 1) < \pi^1(u, h')$, which contradicts stability.

Let’s move to the “if” part. Suppose we have $\pi^2(1, 1) < \pi^1(0, \frac{R^1-1}{R^1}) \wedge R^1 \geq 1$ and $M^1 < 1$. From Equation (4) follows that $M^2 > 0$. Hence $v^c < 1$ and there exists a positive measure agents with $v < 1$ who will join sector two. Clearly, for any such agent $w^2(v) < \pi^2(1, 1)$. Also, as $M^2 > 0$ it has to be the case that there exists a positive measure of firms with $(h, 1)$ such that $\pi^1(u, h) \geq \pi^1(u, \frac{R^1-1}{R^1})$ and $r(h, 1) = 0$. Therefore for the agents and firms in question we have

$$w^2(v) + r(h, 1) < \pi^2(1, 1) < \pi^1(0, \frac{R^1-1}{R^1}) \leq \pi^1(u, h),$$

which contradicts stability and concludes the proof. \square

Proof of Lemma 2. This follows trivially for $R^1 + R^2 < 1$, as then $1 - M^1 - M^2 > 0$ and thus $C_1 = C_2 = 0$, by Lemma 1. Therefore I will focus on the cases where $R^1 + R^2 \geq 1$ and thus $C(u^c, v^c) = 0$. This implies that $\min\{u^c, v^c\} = 0$. Suppose that $u^c = 0$ and $w^1(u^c) > w^2(v^c)$; as the payoff in the second sector is continuous and condition (d) ensures $v^c < 1$, there has to exist some $\epsilon > 0$ such that for any $v \in [v^c, v^c + \epsilon]$ we have $w^1(u^c) > w^2(v)$, which means that none of these agents will join sector two – which contradicts v^c ’s definition.

Now, suppose that $w^1(u^c) < w^2(v^c)$; suppose further that $v^c > 0$. For any $\epsilon > 0$ the mass of agents with $(u, v) \in [0, \epsilon] \times [v^c - \epsilon, v^c]$ is strictly positive – as $C_{uv}(u, v) > 0$ for all (u, v) – and all agents in this set will be working in sector one. However, by continuity of surplus and wage functions it has to be the case that there exists a small enough ϵ that for all agents in this set we have:

$$w^1(u) + \pi^2(v^c, \frac{R^2 - M^2 + v^c}{R^2}) - w^2(v^c) < \pi^2(v, \frac{R^2 - M^2 + v^c}{R^2}),$$

which contradicts stability. Now suppose that $v^c = 0$ as well – there has to exist some $\epsilon > 0$ such that for any $u \in [0, \epsilon]$ we have $w^1(u) < w^2(v^c)$, which contradicts the definition of u^c .

The proof for the case when $v^c = 0$ is analogous. \square

Proof of Lemma 3. I start with sector two. The probability that an agent with talent $V = v$ chooses sector two is $\Pr(\theta(U, v) = 2|v)$. For $v \in [v^c, v^*]$ the probability that a sector two agent has ability lower than v is:

$$G^2(v) = \int_{v^c}^v \frac{\Pr(\theta(U, v) = 2|r)}{R^2} dr,$$

as V 's marginal distribution is standard uniform. Consider some arbitrary agent with $v \in (v^c, v^*]$. Such an agent will be in sector 2 as long as $U \leq \psi(v)$, which implies that $\Pr(\theta(U, v) = 2|v) = C_v(\psi(v), v)^{43}$. Then, the Lemma follows from definitions of v^c and v^* . Sector one follows from analogous reasoning. \square

Proof of Theorem 1. I start by reducing the set of equations and inequalities (6)-(14), which will be henceforth referred to as *the original set*. Consider any $r \in [v^c, v^*]$; then, by differentiating $C(\psi(r), r)$ rearranging and integrating from v^c to v , we arrive at $M^1 G^1(\psi(v)) + M^2 G^2(v) = C(\psi(v), v) - C(u^c, v^c)$, which, rearranged, gives:

$$P^1(\psi(v)) = \frac{1}{R^1} \left[R^1 - 1 + C(\psi(v), v) + R^2(1 - P^2(v)) \right]. \quad (17)$$

An analogous reasoning yields $M^1 G^1(u) + M^2 G^2(\psi^{-1}(u)) = C(u, \psi^{-1}(u)) - C(u^c, v^c)$. This, (9) and (10) imply that $\psi(v^*) = u^*$ (follows also from definitions of u^* and v^* .)

Denote the total mass of matched agents, as $M = M^1 + M^2$; this allows us to write M^2 as $M - M^1$. Using this fact, differentiating Equation (6), dividing both sides by $\pi_u^1\left(\psi(v), \frac{1}{R^1} \int_{u^c}^{\psi(v)} C_u(r, \psi^{-1}(r)) dt\right)$, using Equation (17) and then integrating from v^c to v (and remembering that $\psi(v^c) = u^c$) we get:

$$\psi(v) = u^c + \int_{v^c}^v \frac{\pi_v^2\left(t, \frac{1}{R^2} (R^2 - M^2 + \int_{v^c}^t C_v(\psi(r), r) dr)\right)}{\pi_u^1\left(\psi(t), \frac{1}{R^1} (R^1 - 1 + M^2 + C(\psi(t), t) - \int_{v^c}^t C_v(\psi(r), r) dr)\right)} dt.$$

This equation still depends on $\psi(\cdot)$, u^c , v^c and indirectly on v^* , so we have not really reduced the system yet. We will get rid of v^* by extending the functions

⁴³It doesn't matter whether $\psi(v) \geq u$ holds strictly, as the probability of $\psi(v) = u$ is 0 anyway.

$C(\bullet)$, $\pi^1(\bullet)$ and $\pi^2(\bullet)$ in a way that allows us to define an extended function $\psi^e(\cdot)$, which uniquely determines $\psi(\cdot)$. The extended functions $C^e(\bullet)$, $\pi^{1e}(\bullet)$ and $\pi^{2e}(\bullet)$ are defined as follows: (1) $C^e : [0, 1 + B] \times [0, 1] \rightarrow [0, 1]$

$$C^e(u, v) = \begin{cases} C(u, v) & \text{for } (u, v) \in [0, 1] \times [0, 1] \\ v & \text{for } (u, v) \in (1, 1 + B] \times [0, 1], \end{cases}$$

(2): $\pi^{1e}(u, h) : [0, 1 + B] \times [0, \frac{1+R^1}{R^1}] \rightarrow \mathbf{R}^+$:

$$\pi^{1e}(u, h) = \begin{cases} \pi^1(u, h) & \text{for } (u, h) \in [0, 1]^2 \\ \pi^1(1, h) + (u - 1)\pi_u^1(1, h) & \text{for } (u, h) \in (1, B] \times [0, 1], \\ \pi^1(u, 1) & \text{for } (u, h) \in [0, 1] \times (1, \frac{1+R^1}{R^1}], \\ \pi^1(1, 1) + (u - 1)\pi_u^1(1, 1) & \text{for } (u, h) \in (1, B] \times (1, \frac{1+R^1}{R^1}], \end{cases}$$

(3): $\pi^{2e}(v, h) : [0, 1] \times [0, 1 + \frac{1}{R^2}] \rightarrow \mathbf{R}^+$:

$$\pi^{2e}(v, h) = \begin{cases} \pi^2(v, h) & \text{for } (u, h) \in [0, 1]^2 \\ \pi^2(v, 1) & \text{for } (u, h) \in [0, 1] \times (1, 1 + \frac{1}{R^2}], \end{cases}$$

where $B = \frac{\max \pi_v^2}{\min \pi_u^1}$. The idea behind these extensions is to get functions that will be defined also for $\psi^e(v) > 1$ and such that $C^e(\cdot, v)$, $C_v^e(\cdot, v)$, $\pi_u^{1e}(\cdot, \cdot)$ and $\pi_v^{2e}(v, \cdot)$ are Lipschitz continuous⁴⁴; denote their Lipschitz-constants as L^1, L^2, L^3, L^4 and L^5 respectively.

⁴⁴ We will do this in detail for $C_v^e(u, v)$ – the reasoning for the other two is analogous. $C_v^e(u, v) : [0, 1 + B] \times [0, 1] \rightarrow [0, 1]$:

$$C_v^e(u, v) = \begin{cases} C_v(u, v) & \text{for } (u, v) \in [0, 1] \times [0, 1] \\ 1 & \text{for } (u, v) \in (1, 1 + B] \times [0, 1], \end{cases}$$

is clearly continuous in u . It is equally easy to see that the function $C_v^e(\cdot, v)$ is differentiable almost everywhere and its derivative is Lebesgue integrable. It is also the case that for any $(u, v) \in (1, 1 + B] \times [0, 1]$ we have:

$$C_v^e(a, v) + \int_a^1 C_{uv}^e(r, v) dr + \int_1^u 0 dr = 1,$$

which means that $C_v^e(\cdot, v)$ is absolutely continuous. Moreover, as $C^e(\bullet)$ is twice continuously differentiable and any continuous function defined on a compact set is bounded it follows that $C_v^e(\cdot, v)$ is essentially bounded; and a differentiable almost everywhere, absolutely continuous function with an essentially bounded derivative is Lipschitz-continuous.

Now we can define the extended function $\psi^e(v) : [v^c, 1] \in [u^c, 1 + B]$:

$$\psi^e(v) = u^c + \int_{v^c}^v \frac{\pi_v^{2e} \left(t, \frac{R^2 - M^2 + \int_{v^c}^t C_v^e(\psi(r), r) dr}{R^2} \right)}{\pi_u^{1e} \left(\psi^e(t), \frac{R^1 - 1 + M^2 + C^e(\psi^e(t), t) - \int_{v^c}^t C_v^e(\psi^e(r), r) dr}{R^1} \right)} dt, \quad (18)$$

which together with:

$$M = \min\{R^1 + R^2, 1\} \quad (19)$$

$$1 - M = C^e(u^c, v^c), \quad (20)$$

$$M^2 = \int_{v^c}^1 C_v^e(\psi(r), r) dr, \quad (21)$$

$$v^* = \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}, \quad (22)$$

$$u^* = \psi^e(v^*) \quad (23)$$

$$M^2 \in \Theta(M) \quad (24)$$

and (13) to (14) rewritten in terms of the extended functions, constitute the *modified set* of equations (where $\Theta(M) = [\max\{0, M - R^1\}, \min\{1, R^2\}]$).

Lemma 4. The relation between the original and the modified set is as follows: (a) if ψ^e solves the modified set then its restriction to $[v^c, \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}]$ solves the original one and (b) if a function $\psi(v) : [v^c, v^*] \rightarrow [u^c, u^*]$ solves the original set then we can always find its extension $\psi^e(v) : [v^c, 1] \rightarrow [u^c, 1 + B]$ that solves the modified one.

Proof. Note that if $v^* = \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}$, then $\max\{\psi^e(v^*), v^*\} = 1$, as required. For $v > v^*$, we have $\psi^e(v) > 1$ and thus $C_v^e(\psi(v), v) = 1$, which shows that Equation (21) is equivalent to Equation (9). For $v \leq v^*$ we have that

$$\int_{v^c}^v C_v^e(\psi(r), r) dr \leq M^2,$$

$$C(\psi^e(v), v) - \int_{v^c}^v C_v^e(\psi(r), r) dr \leq M - M^2,$$

which means that the original and extended $C(\bullet)$, $C_v(\bullet)$, $\pi_u^1(\bullet)$ and $\pi_v^2(\bullet)$ are identical; and thus if (18) is met, (6) has to be met too. The equivalence of Equations (23) and (9) follows from the discussion following Equation (17). The equivalence of all other equations is trivial.

Claim (b) is trivial for $\psi(v^*) = 1$, as then ψ and ψ^e coincide. For $\psi(v^*) < 1$ I

claim that

$$\psi^e(v) = \begin{cases} \psi(v) & \text{for } v \in [v^c, v^*] \\ 1 + \int_{v^*}^v \frac{\pi_v^2(t, \frac{1}{R^2}(R^2+1-v))}{\pi_u^1(1,1)} dt & \text{for } v > v^* \end{cases}$$

solves (18)-(24) (and ψ is clearly its restriction for $v \in [v^c, v^*]$). Equation (18) is trivially met for $v \leq v^*$ and to see that it is met for $v > v^*$ it suffices to substitute for π^{1e} , π^{2e} , $C^e(u, v)$, $C_v^e(u, v)$ and use the fact that $\psi^e(v^*) = u^* = 1$. All the other equations are met trivially. \square

Thus, if the solution to the modified set exists and is unique, then the solution to the original set also exists and is unique. Now we will focus on showing that the former is indeed the case. Define the set:

$$K = \{d \in C[0, 1] : |d(v) - 1| \leq 1 + B\},$$

where $C[0, 1]$ is the set of all continuous functions that map from $[0, 1]$. The constant function $d(v) = 1$ lies in K and hence the set is non-empty. Define a norm, $\|\cdot\|_\lambda$ on $C[0, 1]$:

$$\|h\|_\lambda = \sup_{[0,1]} e^{-\lambda v} |h(v)|,$$

where λ is some weakly positive number. K is a complete metric space for this norm.⁴⁵

Endow the sets $[0, 1]^2$ and $\Theta(M)$ with the Euclidean norm and define a mapping $T : K \times [0, 1]^2 \times \Theta(M) \rightarrow K$:

$$(Td)(v, v^c, u^c, M^2) = \begin{cases} u^c & \text{for } v < v^c \\ u^c + \int_{v^c}^v \frac{\pi_v^2(t, \frac{R^2 - M^2 + \int_{v^c}^t C_v^e(d(r), r) dr}{R^2})}{\pi_u^1(d(t), \frac{R^1 - 1 + M^2 + C^e(d(t), t) - \int_{v^c}^t C_v^e(d(r), r) dr}{R^1})} dt & \text{for } v \geq v^c. \end{cases}$$

Note that this map is well-defined, as for any $v^c \in [0, 1]$ and $d \in K$:

$$\begin{aligned} \frac{R^2 - M^2 + \int_{v^c}^t C_v^e(d(r), r) dr}{R^2} &\leq \int_{v^c}^t \frac{R^2 + 1}{R^2} dr \leq \frac{1}{R^2} + 1 \\ \frac{R^1 - 1 + M^2 + C(d(t), t) - \int_{v^c}^t C_v^e(d(r), r) dr}{R^1} &\leq \frac{R^1 + C(d(t), t)}{R^1} \leq \frac{1}{R^1} + 1; \end{aligned}$$

⁴⁵If we endowed K with the sub-norm, then K would be a closed subspace of $C[0, 1]$; since $C[0, 1]$ is complete in the sub-norm, so is K . And it was shown by Bielecki (1956) that the $\|\cdot\|_\lambda$ norm is equivalent to the sup-norm for any $C[a, b]$ – and thus if K is a complete metric space for the sub-norm it is also a complete metric space for $\|\cdot\|_\lambda$.

and that it is continuous in v , v^c , u^c and M^2 . It is also the case that for $v \geq v^c$:

$$|[(Td)(v, v^c, u^c, M^2) - 1] \leq \int_{v^c}^v B dt + |u^c - 1| \leq 1 + B,$$

and for $v < v^c$:

$$|[(Td)(v, v^c, u^c, M^2) - 1] \leq |u^c - 1| \leq 1 + B,$$

so indeed $T(K) \subset K$. Finally, it should be clear that for any (v^c, u^c, M^2) the restriction of any fixed point of $(Td)(\bullet)$ to $[v^c, 1]$ gives us the solution to (18) and that any solution to (18) can be easily extended into a fixed point of $(Td)(\bullet)$. Therefore, it suffices to show that there exists such a λ that for any $(v^c, u^c, M^2) \in [0, 1]^2 \times \Theta(M)$, $Td(\bullet)$ is a contraction wrt to the norm $\|\cdot\|_\lambda$ to show that (18) has a unique solution for any feasible (u^c, v^c, M^2) .

Let us drop (v^c, u^c, M^2) from the definition of the map (remembering that we are keeping them constant) and enhance our notation by new maps: $M^2 : [v^c, 1] \times K \rightarrow [0, 1]$, $P^2 : [v^c, 1] \times K \rightarrow [0, 1 + \frac{1}{R^2}]$ and $P^1 : [0, B] \times K \rightarrow [0, 1 + \frac{1}{R^1}]$:

$$\begin{aligned} (M^2 d)(v) &= \int_{v^c}^v C_v^e(d(r), r) dr, \\ (P^2 d)(v) &= \frac{R^2 - M^2 + (M^1 d)(v)}{R^2}, \\ (P^1 d)(d(v)) &= \frac{R^1 - 1 + M^2 + C^e(d(v), v) - (M^2 d)(v)}{R^1}. \end{aligned}$$

Take any any $t \geq v^c$ and any $d_1, d_2 \in S$ and for any map $(fd)(t)$ denote $(fd_1)(t) - (fd_2)(t)$ as $\Delta_d(fd)(t)$. Then we have:

$$\begin{aligned} |\Delta_d(M^2 d)(t)| &= \left| \int_{v^c}^t C_v^e(d_1(r), r) - C_v^e(d_2(r), r) dr \right| \tag{25} \\ &\leq \int_{v^c}^t |C_v^e(d_1(r), r) - C_v^e(d_2(r), r)| dr \leq \int_{v^c}^t L_2 |d_1(r) - d_2(r)| dr \\ &= L_2 \int_{v^c}^t e^{\lambda r} e^{-\lambda r} |d_1(r) - d_2(r)| dr \leq L_2 \|d_1 - d_2\|_\lambda \int_{v^c}^t e^{\lambda r} dr \\ &= \frac{L_2}{\lambda} \|d_1 - d_2\|_\lambda (e^{\lambda t} - e^{\lambda v^c}) \leq \frac{L_2}{\lambda} \|d_1 - d_2\|_\lambda e^{\lambda t}, \end{aligned}$$

which can be used to establish:

$$|\Delta_d(P^2 d)(t)| \leq \frac{L_2}{\lambda R^2} \|d_1 - d_2\|_\lambda e^{\lambda t} \tag{26}$$

$$\begin{aligned}
|(P^1 d_1)(d_1(t)) - (P^1 d_2)(d_2(t))| &= \left| \frac{C^e(d_1(v), v) - C^e(d_2(v), v) - \Delta_d(M^2 d)(v)}{R^1} \right| \quad (27) \\
&\leq \frac{1}{R^1} (|C^e(d_1(v), v) - C^e(d_2(v), v)| + |\Delta_d(M^2 d)(v)|) \\
&\leq \frac{L_2}{\lambda R^1} \|d_1 - d_2\|_\lambda e^{\lambda t} + \frac{L^1}{R^1} |d_1(t) - d_2(t)|.
\end{aligned}$$

Denote $\sup \pi_v^2(v, h) = L_6$, $\inf \pi_u^1(u, h) = L_7$ and note that continuity of π_u^1 and π_v^2 and the fact that $\pi_u^1 > 0$ imply that both L_6 and L_7 are finite. Using all this, we can write, for any $v \geq v^c$ and any $d_1, d_2 \in S$:

$$\begin{aligned}
|\Delta_d(Td)(v)| &= \left| \int_{v^c}^v \frac{\pi_v^{2e}(t, (P^2 d_1)(t))}{\pi_u^{1e}(d_1(r), (P^1 d_1)(d_1(t)))} - \frac{\pi_v^{2e}(t, (P^2 d_2)(t))}{\pi_u^{1e}(d_2(r), (P^1 d_2)(d_2(t)))} dt \right| \\
&\leq \int_{v^c}^v \left| \frac{\pi_v^{2e}(t, (P^2 d_1)(t))}{\pi_u^{1e}(d_1(r), (P^1 d_1)(d_1(t)))} - \frac{\pi_v^{2e}(t, (P^2 d_2)(v,))}{\pi_u^{1e}(d_1(r), (P^1 d_1)(d_1(t)))} \right. \\
&\quad \left. + \frac{\pi_v^{2e}(t, (P^2 d_2)(t))}{\pi_u^{1e}(d_1(r), (P^1 d_1)(d_1(t)))} - \frac{\pi_v^{2e}(t, (P^2 d_2)(t))}{\pi_u^{1e}(d_2(r), (P^1 d_2)(d_2(t)))} \right| dt \\
&\leq \int_{v^c}^v \frac{|\pi_v^{2e}(t, (P^2 d_1)(t)) - \pi_v^{2e}(t, (P^2 d_2)(t))|}{L_7} \\
&\quad + L_6 \left| \frac{\pi_u^{1e}(d_1(r), (P^1 d_1)(d_1(t))) - \pi_u^{1e}(d_2(r), (P^1 d_2)(d_2(t)))}{\pi_u^{1e}(d_1(r), (P^1 d_1)(d_2(t))) \pi_u^{1e}(d_2(r), (P^1 d_2)(d_2(t)))} \right| dt \\
&\leq \int_{v^c}^v \frac{L_5}{L_7} |\Delta_d(P^2 d)(t)| \\
&\quad + \frac{L_6}{L_7^2} [|\pi_u^{1e}(d_1(r), (P^1 d_1)(d_1(t))) - \pi_u^{1e}(d_2(t), (P^1 d_1)(d_1(t)))| \\
&\quad + |\pi_u^{1e}(d_2(t), (P^1 d_1)(d_1(t))) - \pi_u^{1e}(d_2(r), (P^1 d_2)(d_2(t)))|] dt \\
&\leq \int_{v^c}^v \frac{L_5 L_2}{\lambda L_7 R^2} \|d_1 - d_2\|_\lambda e^{\lambda(t-v^c)} + \frac{L_3 L_6}{L_7^2} |d_1(t) - d_2(t)| \\
&\quad + \frac{L_4 L_6}{L_7^2} |(P^1 d_1)(d_1(t)) - (P^1 d_2)(d_2(t))| dt \\
&\leq \frac{L_5 L_2}{\lambda^2 L_7 R^2} \|d_1 - d_2\|_\lambda e^{\lambda v} + \frac{L_3 L_6}{\lambda L_7^2} \|d_1 - d_2\|_\lambda e^{\lambda v} \\
&\quad + \int_{v^c}^v \frac{L_4 L_6}{L_7^2} \left(\frac{L_2}{\lambda R^1} \|d_1 - d_2\|_\lambda e^{\lambda(t-v^c)} + \frac{L_1}{R^1} |d_1(t) - d_2(t)| \right) dt \\
&\leq \frac{1}{\lambda} \|d_1 - d_2\|_\lambda e^{\lambda v} \left[\frac{L_5 L_2}{\lambda L_7 R^2} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2} \left(\frac{L_2}{\lambda R^1} + \frac{L_1}{R^1} \right) \right]
\end{aligned}$$

Now, for $v < v^c$ this has to hold as well, as then $|(T d_1)(v) - T(d_2)(v)| = 0$; therefore, for any $v \in [0, 1]$ we have that:

$$|\Delta_d(Td)(v)| \leq \frac{1}{\lambda} \|d_1 - d_2\|_\lambda e^{\lambda v} \left[\frac{L_5 L_2}{\lambda L_7 R^2} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2} \left(\frac{L_2}{\lambda R^1} + \frac{L_1}{R^1} \right) \right].$$

Dividing both sides of that by $e^{\lambda v}$ and then taking sup on both sides we get:

$$\|(Td_1)(t) - T(d_2)(t)\|_\lambda \leq \frac{1}{\lambda} \|d_1 - d_2\|_\lambda \left[\frac{L_5 L_2}{\lambda L_7 R^2} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2} \left(\frac{L_2}{\lambda R^1} + \frac{L_1}{R^1} \right) \right]. \quad (28)$$

Therefore, there has to exist a high enough λ for which our map $(Td)(v)$ is a contraction in the metric space $(S, \|\cdot\|_\lambda)$ – which, by Banach’s Fixed-Point Theorem means that $(Td)(v)$ has a unique fixed point, which in turn means that Equation (18) has a single solution for any given $(v^c, u^c, M^2) \in [0, 1]^2 \times \Theta(M)$. Note that Equation (28) does not depend on (v^c, u^c, M^2) – and thus, by standard results (see e.g. Hasselblatt and Katok, 2003, p. 68) it follows that as $(Td)(v, v^c, u^c, M^2)$ is continuous in v^c , u^c and M^2 the fixed point – and thus the solution of (18) – is continuous in them as well.

Denote the fixed point of $(Td)(\cdot, v^c, u^c, M^2)$ as $d^*(\cdot, v^c, u^c, M^2)$ – then the following result holds:

Lemma 5. The function $d^*(\cdot, v^c, u^c, M^2)$ is weakly decreasing in v^c and M^2 and weakly increasing in u^c for all v ’s. Moreover, for some v ’s, $d^*(\cdot, v^c, u^c, M^2)$ is strictly decreasing in v^c and M^2 (strictly increasing in u^c).

Proof. I start with the claims regarding $d(v, \cdot, u^c, M^2)$ and suppress u^c and M^2 from notation for that part of the proof. Take any $v_2^c > v_1^c \in [0, 1]$, denote $d^*(v, v_2^c) - d^*(v, v_1^c)$ as $\Delta_{v^c} d^*(v, v^c)$ and define:

$$\begin{aligned} M^2(v, v^c) &= \int_{v^c}^v C_v(d^*(r, v^c), r) dr, \\ P^2(v, v^c) &= \frac{R^2 - M^2 + M^2(v, v^c)}{R^2}, \\ P^1(d^*(v, v^c), v^c) &= \frac{R^1 - 1 + M^2 + C(d^*(v, v^c), r) - M^2(v, v^c)}{R^1}. \end{aligned}$$

Then for any $v \geq v_2^c$ we have:

$$\begin{aligned} \Delta_{v^c} d^*(v, v^c) &= \Delta_{v^c} d^*(v_2^c, v^c) \\ &+ \int_{v_2^c}^v \frac{\pi_v^{2e}(t, P^2(v_2^c, t))}{\pi_u^{1e}(d^*(t, v_2^c), P^1(v_2^c, d^*(t, v_2^c)))} - \frac{\pi_v^{2e}(t, P^2(v_1^c, t))}{\pi_u^{1e}(d^*(t, v_1^c), P^1(v_1^c, d^*(t, v_1^c)))} dt. \end{aligned}$$

It is trivial that for any $v \in [v_1^c, v_2^c)$ we have $\Delta_{v^c} d^*(v, v^c) < 0$, which proves the second (strict) part of this claim. Thus, we only need to show now that $\Delta_{v^c} d^*(v, v^c) \leq 0$ for all $v \in [v_2^c, 1]$. Suppose not. Then the set $\Omega^{gen} = \{v \in [v_2^c, 1] : \Delta_{v^c} d^*(v, v^c) > 0\}$ has to be non-empty. Then we have that for $v^g = \inf \Omega^{gen}$,

$\Delta_{v^c} d^*(v^g, v^c) = 0$ and $\Delta_{v^c} d_v^*(v^g, v^c) > 0$. The sign of $\Delta_{v^c} d_v^*(v^g, v^c)$ depends only on the signs of⁴⁶:

$$\pi_v^{2e}(v^g, P^2(v_2^c, v^g)) - \pi_v^{2e}(v^g, P^2(v_1^c, v^g))$$

and

$$\pi_u^{1e}(d^*(v^g, v_1^c), P^1(v_1^c, d^*(v^g, v_1^c))) - \pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c))).$$

However, as $\Delta_{v^c} d^*(v^g, v^c) = 0$ and both surplus functions are weakly supermodular, these in turn depend only on the sign of $M^2(v_2^c, v^g) - M^2(v_1^c, v^g)$. As for any $v \leq v^g$ it was the case that $\Delta_{v^c} d^*(v^g, v^c) \leq 0$ and $v_2^c \geq v_1^c$, it follows that: $M^2(v_2^c, v^g) - M^2(v_1^c, v^g) \leq 0$ and thus:

$$\Delta_{v^c} d_v^*(v, v^c) \leq 0,$$

which means that Ω^{gen} has to be empty and proves our first claim.

The proof for M^2 is analogous⁴⁷. For u^c , note that for a change in u^c , $\Delta_{u^c} d^*(v^c, u^c)$ is positive. The subsequent reasoning is analogous, but with opposite signs (the strict decreasingness follows from $\Delta_{u^c} d^*(v^c, u^c) < 0$ and continuity). \square

Everything I derived so far applies both for cases with abundant and scarce jobs. From now on, however, I will consider those cases separately.

Scarce jobs If $R^1 + R^2 < 1$, then $M = R^1 + R^2$, which reduces (24) to $M^2 = R^2$ and gives $C(u^c, v^c) = 1 - R^1 - R^2 > 0$. For $(u, v) > 0$, $C(\bullet)$ is strictly increasing in both parameters, which allows us to define u^c as a strictly decreasing, continuous function of v^c . Define \underline{v} as $u^c(\underline{v}) = 1$ and note that, as $u^c \in [0, 1]$, Equation (20) shrinks the range of feasible v^c 's to $[\underline{v}, 1]$. Hence, $d^*(v, v^c, u^c, M^2)$ depends only on

⁴⁶To see this, note that:

$$\begin{aligned} \Delta_{v^c} d_v^*(v^g, v^c) &= \frac{\pi_v^{2e}(v^g, P^2(v_2^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c)))} - \frac{\pi_v^{2e}(v^g, P^2(v_1^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_1^c), P^1(v_1^c, d^*(v^g, v_1^c)))} \\ &= \frac{\pi_v^{2e}(v^g, P^2(v_2^c, v^g)) - \pi_v^{2e}(v^g, P^2(v_1^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c)))} \\ &\quad + \pi_v^{2e}(v^g, P^2(v_1^c, v^g)) \frac{\pi_u^{1e}(d^*(v^g, v_1^c), P^1(v_1^c, d^*(v^g, v_1^c))) - \pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c)))}{\pi_u^{1e}(d^*(v^g, v_1^c), P^1(v_1^c, d^*(v^g, v_1^c))) \pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c)))}. \end{aligned}$$

⁴⁷For $v = v^c$ we have $\Delta_M d^*(v, M^2) = 0$ and $\Delta_M d_v^*(v, M^2) < 0$. The sign of $\Delta_M d_v^*(v^g, M^2)$ depends on $M_1^2 - M_2^2 < 0$ and the difference in $M^2(v, M^2)$, which is weakly negative for the same reasons as above. Thus, $\Delta_M d_v^*(v^c, M^2) \leq 0$, which implies that $d^*(v, a, \cdot)$ will never strictly increase.

v and v^c and is decreasing and continuous in v^c – I will denote it as $d^*(v, v^c)$ from now on. Thus, the modified system of equations reduces to:

$$R^2 = \int_{v^c}^1 C_v^e(d^*(r, v^c), r) dr.$$

The RHS is continuous in v^c , as $d^*(v, v^c)$ is continuous in v^c . For $v^c = \underline{v}$, we have $d^*(v, v^c) \geq 1$ regardless of v and therefore $\int_0^1 C_v^e(d^*(r, v^c), r) dr = 1$, whereas for $v^c = 1$, $\int_1^1 C_v^e(d^*(r, v^c), r) dr = 0$; thus, a solution to (21) (given $R^2 \in (0, 1)$) exists. It is unique, as $d^*(v, \cdot)$ is weakly decreasing for all and strictly decreasing for some v and thus the RHS crosses R^2 only once from above.

Abundant jobs If $R^1 + R^2 \geq 1$, then $M = 1$ and thus $C(u^c, v^c) = 0$. Hence, $\min\{u^c, v^c\} = 0$ and I cannot define u^c as a function of v^c , as there is a continuum of v^c 's for which $C(0, v^c) = 0$. I address this by defining the set $\Gamma^c = \{(u^c, v^c) : \min\{u^c, v^c\} = 0\}$, a new variable $a \in [-1, 1]$ and writing u^c and v^c as:

$$u^c(a) = \begin{cases} -a & \text{for } a \leq 0, \\ 0 & \text{for } a > 0, \end{cases} \quad v^c(a) = \begin{cases} 0 & \text{for } a \leq 0, \\ a & \text{for } a > 0. \end{cases}$$

For any a , $(u^c(a), v^c(a)) \in \Gamma^c$ and for any $(u^c, v^c) \in \Gamma^c$ there exists a unique a , such that $(u^c(a), v^c(a)) = (u^c, v^c)$. Thus, if there exists a unique a that solves Equation (21), there also exists a unique (u^c, v^c) that solves it. Moreover, $v^c(a)$ is continuous and increasing, and $u^c(a)$ is continuous and decreasing. Therefore the function $d^*(v, a, M^2) = d^*(v, v^c(a), u^c(a), M^2)$ is continuous and decreasing (strictly for some v 's) in a . Thus, I can write Equation (21) as:

$$M^2 = \begin{cases} \int_0^1 C_v^e(d^*(r, a, M^2), r) dr & \text{for } a < 0, \\ \int_a^1 C_v^e(d^*(r, a, M^2), r) dr & \text{for } a \geq 0. \end{cases}$$

The RHS is continuous in a , as $d^*(v, a, M^2)$ is continuous in a . For $a = -1$, we have $\int_0^1 C_v^e(d^*(r, a, M^2), r) dr = 1$; for $a = 1$, we have $\int_a^1 C_v^e(d^*(r, a, M^2), r) dr = 0$; thus, a solution to (21) (given $M^2 \in \Theta(1)$) exists. It is unique, as $d^*(v, \cdot, M^2)$ is weakly decreasing for all and strictly decreasing for some v and thus the RHS crosses M^2 only once from above.

As $d^*(v, \cdot, \cdot)$ is continuous, $a(M^2)$ is continuous as well. It is strictly decreasing in M^2 , as the LHS is strictly increasing in M^2 and the RHS is weakly decreasing in M^2 and strictly decreasing in a ; thus, if M^2 increases, Equation (21) is met only if

a decreases. As $a(M^2)$ is unique and a defines uniquely (u^c, v^c) , there exist unique $u^c(M^2)$ and $v^c(M^2)$; the former is non-decreasing and the latter non-increasing; and for any $M_2^2 > M_1^2$ we have that $u^c(M_2^2) > u^c(M_1^2)$ or $v^c(M_2^2) < v^c(M_1^2)$.

The modified set reduces to:

$$M^2 > 1 - R^1 \Rightarrow \pi^1\left(u^c(M^2), \frac{R^1 - 1 + M^2}{R^1}\right) \leq \pi^2\left(v^c(M^2), \frac{R^2 - M^2}{R^2}\right) \quad (29)$$

$$M^2 < R^2 \Rightarrow \pi^1\left(u^c(M^2), \frac{R^1 - 1 + M^2}{R^1}\right) \geq \pi^2\left(v^c(M^2), \frac{R^2 - M^2}{R^2}\right) \quad (30)$$

$$M^2 \in \Theta(1). \quad (31)$$

Note that $u^c(0) = 0$, $v^c(0) = 1$, $u^c(1) = 1$ and $v^c(1) = 0$. Condition (29) -(30) will be trivially met if there exists some $M^2 \in \Theta(1)$ such that:

$$\pi^1\left(u^c(M^2), \frac{R^1 - 1 + M^2}{R^1}\right) = \pi^2\left(v^c(M^2), \frac{R^2 - M^2}{R^2}\right).$$

If there is no such M^2 , then it has to be the case that either (a) $LHS > RHS$ for all $M^2 \in \Theta(1)$ or (b) $RHS > LHS$ for all $M^2 \in \Theta(1)$. However, (a) is possible only if $\max\{0, 1 - R^1\} = 1 - R^1$, as $LHS > RHS$ for $M^2 = 0$ violates condition (d). And for $M^2 = 1 - R^1$, $LHS > RHS$ meets (29) -(30), as the first inequality doesn't have to hold. For similar reasons, (b) is possible only if $\min\{1, R^2\} = R^2$, in which case $RHS > LHS$ meets (29) -(30). Thus, existence of a solution to (29) -(30) follows. Hence, there exists a solution to the modified and original sets.

For uniqueness, remember that $d^*(v, a(M^2), M^2)$ is unique and, thus, it suffices to show that the solution to (29) -(30) is unique. Denote the set of all $M^2 \in \Theta(1)$ that meet (29) -(30) as Ω^M . Consider $\min \Omega^M = M_1^2$. Note that M_1^2 exists as Ω^M is non-empty and $\pi^1(\cdot, \cdot)$, $\pi^2(\cdot, \cdot)$, $v^c(\cdot)$ and $u^c(\cdot)$ are continuous. Suppose $M_1^2 = \min\{1, R^2\}$ – then the solution is unique. Now suppose that $M_1^2 < \min\{1, R^2\}$, which implies that for any $M_2^2 \in \Omega^M$ such that $M_2^2 > M_1^2$ we need to have:

$$\pi^1\left(u^c(M_2^2), \frac{R^1 - 1 + M_2^2}{R^1}\right) \leq \pi^2\left(v^c(M_2^2), \frac{R^2 - M_2^2}{R^2}\right)$$

and for M_1^2 we have:

$$\pi^1\left(u^c(M_1^2), \frac{R^1 - 1 + M_1^2}{R^1}\right) \geq \pi^2\left(v^c(M_1^2), \frac{R^2 - M_1^2}{R^2}\right).$$

This is a contradiction, as $\pi_u^1 > 0$, $\pi_h^1 \geq 0$, $\pi_v^2 > 0$, $\pi_h^2 \geq 0$, $u^c(\cdot)$ is weakly increasing, $v^c(\cdot)$ is weakly decreasing and $u^c(M_2^2) > u^c(M_1^2) \vee v^c(M_2^2) < v^c(M_1^2)$. Thus M_2^2 does not exist and M_1^2 is the only element in Ω^M , which completes the proof. \square

B Comparative Statics

To simplify what follows, I first introduce new notation. The difference between the old and new values of any object O is denoted as $\Delta_\rho O$. The greater of the old and new values of O is denoted as $\max O$. Thus, for instance, the change in sector one size is $\Delta_\rho M^1$ and the greater critical ability in sector two is $\max v^c$.

Definition 14. *Vertical differentiation* increases in sector one if $\pi_u^1(u, h, \rho_2) \geq (>)$ $\pi_u^2(u, h, \rho_1)$ for all (u, h) .

Definition 15. The matching problems $(Q(\rho_1), Q(\rho_2))$ have *(strong) impossibility property* if it is impossible that $v^c(\rho_2) < (\leq)v^c(\rho_1)$ and $\Delta_\rho M^2 > (\geq)0$.

Theorem 2. Suppose $(Q(\rho_1), Q(\rho_2))$ exhibit the impossibility property and sector sizes are unchanged. Then $\Delta_\rho \pi_u^1(\bullet) \geq 0$ for all (u, h) results in (i) $P^2(v, \rho_2) \geq P^2(v, \rho_1)$ for all v and (ii) $P^1(u, \rho_2) \leq P^1(u, \rho_1)$ for all u . If the property is strong, then (i) holds strictly for a positive measure of v and (ii) for a positive measure of u . If $\Delta_\rho \pi_u^1(\bullet) > 0$ for all (u, h) , then (iii) $P^2(v, \rho_2) > P^2(v, \rho_1)$ for all $v \in [\max v^c, \max v^*)$ and (iv) $P^1(u, \rho_2) < P^1(u, \rho_1)$ for all $u \in [\max u^c, \max u^*)$.

Proof of Theorem 2. The results for sector two are proved in a series of lemmas and the result for sector one follow easily (details at the end of the proof). But first, I define the following three sets of sector two talent levels:

$$\begin{aligned}\Xi^0 &= \{v \in [\max v^c, \min v^*] : \psi(v, \rho_2) = \psi(v, \rho_1) \wedge P^2(v, \rho_2) \leq P^2(v, \rho_1)\} \\ \Xi^1 &= \{v \in [\max v^c, \min v^*] : \psi(v, \rho_2) \leq \psi(v, \rho_1) \wedge P^2(v, \rho_2) < P^2(v, \rho_1)\} \\ \Xi^2 &= \{v \in [\max v^c, \min v^*] : \psi(v, \rho_2) \leq \psi(v, \rho_1) \wedge P^2(v, \rho_2) \leq P^2(v, \rho_1)\}.\end{aligned}$$

Lemma 6. A (strict) increase in sector one vertical differentiation implies that Ξ^1 (Ξ^2) is empty.

Proof of Lemma 6. Remember that $P_v^2(v) = \frac{\psi_v(v)C_{uv}(\psi(v), v)}{R^2}$. Take any $v_0 \in \Xi^0$. Note that by Equation (17) we have $\Delta_\rho P^1(\psi(v_0, \rho_1)) \geq 0$. Then we have:

$$\begin{aligned}\Delta_\rho w_u^1(\psi(v_0, \rho_1)) &= \Delta_\rho \pi_u^1(\psi(v_0, \rho_1), P^1(\psi(v_0, \rho_1), \rho_2)) \\ &\quad + \int_{P^2(\psi(v_0, \rho_1), \rho_1)}^{P^2(\psi(v_0, \rho_1), \rho_2)} \pi_{uh}^1(\psi(v_0, \rho_1), r, \rho_1) dr \geq (>)0,\end{aligned}$$

as $\Delta_\rho \pi_u^1(u, h) \geq (>)0$ for any (u, h) , $\pi^1(\bullet)$ is supermodular and $\Delta_\rho P^1(\psi(v_0, \rho_1)) \geq$

0. Whereas for v_0 we have:

$$\Delta_\rho w_v^2(v_0) = \int_{P^2(v_0, \rho_1)}^{P^2(v_0, \rho_2)} \pi_{vh}^2(v_0, r) dr \leq 0,$$

as $\pi^2(\bullet)$ is supermodular and $\Delta_\rho P^2(v_0) \leq 0$. By differentiating Equation (5) wrt to v for both ρ_2 and ρ_1 , taking differences and rearranging, we arrive at:

$$\Delta_\rho \psi_v(v_0) = \frac{1}{w_u^1(\psi(v_0, \rho_1), \rho_2)} [\Delta_\rho w_v^2(v_0) - \psi_v(v_0, \rho_1) \Delta_\rho w_u^1(\psi(v_0, \rho_1))],$$

from which follows trivially that $\Delta_\rho \psi_v(v_0) \leq (<)0$.

Take any $v_1 \in \Xi^1$. Suppose that $\Delta_\rho \psi(v) \leq 0$ for all $v \in [v_1, \min v^*]$, which implies that $v^*(\rho_2) > v^*(\rho_1)$. Remember that both for ρ_1 and ρ_2 we need to have $P^2(1) = 1$ and thus $\Delta_\rho P^2(1) = 0$. Therefore:

$$\begin{aligned} 0 &= \Delta_\rho P^2(v_1) + \frac{\int_{v_1}^{v^*(\rho_2)} C_v(\psi(v, \rho_2), v) dv - \int_{v_1}^{v^*(\rho_1)} C_v(\psi(v, \rho_1), v) dv - \Delta_\rho v^*}{R^2} \\ &= \Delta_\rho P^2(v_1, \rho_1) - \int_{v_1}^{v^*(\rho_1)} \int_{\psi(v, \rho_2)}^{\psi(v, \rho_1)} \frac{C_{uv}(s, v)}{R^2} ds dv - \int_{v^*(\rho_2)}^{v^*(\rho_1)} \frac{1 - C_v(\psi(v, \rho_2), v)}{R^2} dv. \end{aligned}$$

Note that as $\psi(v, \rho_2) \leq 1$ it follows that $C_v(\psi(v, \rho_2), v) \leq 1$; hence we have that the two latter terms on the RHS are weakly and the first is strictly negative – contradiction. Therefore there needs to exist some $v \in (v_1, \min v^*]$ such that $\Delta_\rho \psi(v) > 0$ for Ξ^1 to be non-empty. Denote the set of all such v 's as Ξ^3 ; then it follows that $\inf \Xi^3 \in \Xi^{048}$. But under $\Delta_\rho \pi_u^1(u, h) \geq (>)0$ for all (u, h) for any $v \in \Xi^0$, $\Delta_\rho \psi_v(v) \leq (<)0$, which contradicts $v = \inf \Xi^3$. Thus Ξ^1 has to be empty and the result for a weak increase in vertical differentiation holds.

Now consider any $v_2 \in \Xi^2$. Note that under $\Delta_\rho \pi_u^1(u, h) > 0$ for all (u, h) there has to exist some arbitrarily small $\epsilon > 0$ such that $v_2 + \epsilon < \min v^*$ and for any $v \in (v_2; v_2 + \epsilon]$ we have $\Delta_\rho \psi(v) < 0$. This follows from continuity if $\Delta_\rho \psi(v_2) < 0$ and from the fact that if $\Delta_\rho \psi_v(v_2) = 0$ then $v_2 \in \Xi^0$ and $\Delta_\rho \psi_v(v_2) < 0$. Therefore, trivially, $\Delta_\rho P^2(v_2 + \epsilon) < 0$ and thus $v_2 + \epsilon \in \Xi^1$, which means that Ξ^2 has to be empty and concludes the proof. \square

Lemma 7. Suppose Ξ^1 is empty. Consider some $v_e \in [\max v^c, \min v^*]$. Then $\Delta_\rho P^2(v_e) \geq 0$ implies $\Delta_\rho P^2(v_e) \geq 0$ for all $v \in [v_e, \min v^*]$. If Ξ^2 is empty, then additionally $\Delta_\rho P^2(v_e) > 0$ implies $\Delta_\rho P^2(v_e) > 0$ for all $v \in [v_e, \min v^*]$.

Proof. I will start with the first claim. Suppose it is false. Then the set $\Upsilon^1 =$

⁴⁸By continuity of $\Delta_\rho \psi(v)$, which follows from continuity of $\psi(v)$.

$\{v \in [v_e, \min v^*] : \Delta_\rho P^2(v) < 0\}$ has to be non-empty. Take some $v^1 \in \Upsilon^1$ and define $\Upsilon^2 = \{v \in [v_e, v_1] : \Delta_\rho P^2(v) \geq 0\}$. By continuity of $\Delta_\rho P^2(v)$ the point $v^2 = \max \Upsilon^2$ exists and is $< v^1$. Therefore, for any $v \in (v^2, v^1]$ we have $\Delta_\rho P^2(v) < 0$. However, as:

$$\Delta_\rho P^2(v^1) = \Delta_\rho P^2(v^2) + \frac{1}{R^2} \int_{v^2}^{v^1} \int_{\psi(r, \rho_1)}^{\psi(r, \rho_2)} C_{uv}(s, r) ds dr,$$

this implies that there exists some $v_1 \in (v^2, v^1]$ such that $\Delta_\rho \psi(v_1) < 0$ and thus $v_1 \in \Xi^1$ – contradiction.

Let us move to the second claim. Again, suppose it is false. Then the set $\Upsilon^3 = \{v \in [v_e, \min v^*] : \Delta_\rho P^2(v) \leq 0\}$ has to be non-empty; but as $\Delta_\rho P^2(v)$ is continuous in v , the non-emptiness implies that $v^3 = \min \Upsilon^3$ exists. Additionally, $v^3 > v_e$, as $\Delta_\rho P^2(v_e) > 0$. Define a new set $\Upsilon^4 = \{v \in [v_e, v^3] : \Delta_\rho \psi(v) \leq 0\}$ and $v^4 = \max \Upsilon^4$; by definition of v^3 , for any $v < v^3 \wedge v \in \Upsilon^4$ we have that $\Delta_\rho P^2(v) > 0$. As $[v_e, v^3]$ is a compact set and $\Delta_\rho \psi(v)$ is continuous v^4 won't exist only if Υ^4 is empty; but an empty Υ^4 implies that $\Delta_\rho \psi(v) > 0$ for any $v \in [v_e, v^3]$, which in turn means that $\Delta_\rho P^2(v^3) > 0$, which contradicts the definition of v^3 . Therefore v^4 needs to exist. Now suppose that $v^4 < v^3$; then we have $\Delta_\rho P^2(v^4) > 0$ and for any $v \in (v^4, v^3]$, $\Delta_\rho \psi(v) > 0$, which implies that $\Delta_\rho P^2(v^3) > 0$ and also contradicts the definition of v^3 . Therefore it has to be the case that $v^3 = v^4$; but this implies that $\Delta_\rho(\psi(v^3)) \leq 0$ and $\Delta_\rho P^2(v^3) \leq 0$, which contradicts emptiness of Ξ^2 \square

Lemma 8. $\Delta_\rho P^2(\min v^*) \geq 0$ implies that (i) for any $v > \min v^*$ we have $\Delta_\rho P^2(v) \geq 0$ and (ii) for all $v \in [\min v^*, \max v^*]$ we have $\Delta_\rho P^2(v) > 0$.

Proof. Note that $\Delta_\rho P^2(\min v^*) > (\geq) 0$ implies that $v^*(\rho_2) > (\geq) v^*(\rho_1)$ ⁴⁹. Thus, if $\Delta_\rho P^2(\min v^*) = 0$ then $\min v^* = \max v^c$ and the second claim follows trivially. Whereas if $\Delta_\rho P^2(\min v^*) > 0$ then $v^*(\rho_2) > v^*(\rho_1)$ and by the fact that all agents with $v \in (v^*, 1]$ join sector two for sure it follows that for $v \in (v^*(\rho_1), v^*(\rho_2))$ we also have $\Delta_\rho P^2(v) > 0$. Claim (i) for $v > \max v^*$ follows easily from the aforementioned property of v^* . \square

⁴⁹ To see this, denote the ρ_i for which $v^*(\rho_i) = \max v^*$ as ρ_m ; then, as $\Delta_\rho P^2(1) = 0$, we have:

$$0 = \Delta_\rho P^2(\min v^*, \rho_1) + \frac{1}{R^2} \int_{v^*(\rho_2)}^{v^*(\rho_1)} \frac{1 - C_v(\psi(v, \rho_m), v)}{R^2} dv.$$

As $1 - C_v(\psi(v, \rho_m), v) \geq 0$, the fact that $\Delta_\rho P^2(\min v^*) > (\geq) 0$ implies that for this to hold we need $v^*(\rho_2) > (\geq) v^*(\rho_1)$.

Lemma 9. The (strong) impossibility property implies that if $v^c(\rho_2) < (\leq)v^c(\rho_1)$ then $\Delta_\rho P^2(v^c(\rho_1)) > 0$.

Proof. This follows from the fact that $\Delta_\rho v^c < (\leq)0$ implies that $G^2(v^c(\rho_1)) > (\geq)0$, the fact that:

$$\Delta_\rho P^2(v) = \frac{1}{R^2}((G^2(v, \rho_2) - 1)\Delta_\rho M^2 + M^2(\rho_1)\Delta_\rho G^2(v)) \quad (32)$$

and the fact that $v^c(\rho_1) < 1$ and thus $G^2(v, \rho_2) - 1 < 0$. \square

Lemma 10. Empty Ξ^1 and impossibility property jointly imply $\Delta_\rho P^2(\max v^c) \geq 0$. If either the increase in vertical differentiation is strict or the property is strong then this inequality holds strictly.

Proof. Suppose (strong) impossibility property holds. Define a set $\Xi^5 = \{v \in [\max v^c, \max v^*] : \Delta_\rho \psi(v_5) < 0 \wedge \Delta_\rho P^2(v_5) \leq 0\}$. By continuity, there has to exist some some arbitrarily small $\epsilon > 0$ such that $v_5 + \epsilon \in \Xi^1$; thus, by Lemma 6, an increase in vertical differentiation implies that Ξ^5 has to be empty.

If $v^c(\rho_2) < (\leq)v^c(\rho_1)$ – then by Lemma 9 we have $\Delta_\rho P^2(\max v^c) > 0$. If $v^c(\rho_2) \geq v^c(\rho_1)$ and $\max v^c \geq \min v^*$, then – as $v^* > v^c$ – it has to be that $v^*(\rho_2) > v^c(\rho_2) > v^*(\rho_1)$. But as all agents with $v > v^*$ join sector two, this implies $\Delta_\rho P^2(v^c(\rho_2)) > 0$.

Thus, we only need to show the result for $\max v^c < \min v^*$ and $v^c(\rho_2) \geq (>)v^c(\rho_1)$. As $\Delta_\rho M^3 = 0$ we have $C(u^c(\rho_1), v^c(\rho_1)) = C(u^c(\rho_2), v^c(\rho_2))$ and thus $\Delta_\rho v^c \geq (>)0$ implies $\Delta_\rho u^c \leq 0$. As $\psi(v^c) = u^c$ and $\psi(v)$ is strictly increasing for any ρ we have: $\psi(v^c(\rho_2), \rho_1) \geq (>)u^c(\rho_1)$, $u^c(\rho_1) \geq u^c(\rho_2)$ and $u^c(\rho_2) = \psi(v^c(\rho_2), \rho_2)$, which trivially implies that:

$$\Delta_\rho \psi(v^c(\rho_2)) \leq (<)0.$$

If second property holds, then this inequality holds weakly, which together with empty Ξ^1 implies $\Delta_\rho P^2(v^c(\rho_1)) \geq 0$. If the second property is strong, then $\Delta_\rho \psi(v^c(\rho_2)) < 0$, which – as Ξ^5 is empty – implies $\Delta_\rho P^2(\max v^c) > 0$. If Ξ^2 is empty, then we have that $\Delta_\rho \psi(v^c(\rho_2)) \leq 0$ implies $\Delta_\rho P^2(\max v^c) > 0$, which concludes the proof. \square

Lemma 11. Empty Ξ^1 and impossibility properties imply jointly that for any $v < \max v^c$, $\Delta_\rho P^2(v) \geq 0$.

Proof. Suppose $\Delta_\rho v^c < 0$ – then for all $v < \max v^c$ we have that $G^2(v^c(\rho_1)) \geq 0$ and by impossibility property that $\Delta_\rho M^2 \leq 0$. Thus, the claim follows from Equation (32). Now suppose that $\Delta_\rho v^c \geq 0$. This implies that for any $v \leq v^c(\rho_2)$ it is the case that $\Delta_\rho G^2(v^c(\rho_2)) = 0 - G^2(v, \rho_2) \leq 0$ and this expression is decreasing in v . As by Lemma 10 $\Delta_\rho P^2(v^c(\rho_2)) \geq 0$ it follows from Equation (32) that $\Delta_\rho P^2(v) \geq 0$ for all $v < \max v^c$, as required. Note that this implies also that $\Delta_\rho P^2(0, \rho) = -\Delta M^2 \geq 0$. \square

Lemma 12. For all $v \in [\max v^c, \min v^*]$, if $\Delta_\rho P^2(v) \geq (>)0$ then $\Delta_\rho P^1(\psi(v, \rho_2)) \leq (<)0$.

Proof. From Equation (17), Definition 6 and Lemma 3 follows that:

$$\begin{aligned} \Delta_\rho P^2(\psi(v, \rho_2)) = & -\frac{1}{R^1} R^2 \Delta_\rho P^2(v) \\ & + \frac{1}{R^1} \left[\int_{\psi(v, \rho_1)}^{\psi(v, \rho_2)} C_u(r, v) dr - \int_{\psi(v, \rho_1)}^{\psi(v, \rho_2)} C_u(r, \phi(r, \rho_1)) dr \right]. \end{aligned}$$

If $\psi(v, \rho_2) \geq \psi(v, \rho_1)$ then for any $r \in [\psi(v, \rho_1), \psi(v, \rho_2)]$, $\phi(r, \rho_1) \geq v$ and my claim follows. If $\psi(v, \rho_2) < \psi(v, \rho_1)$ then for any $r \in [\psi(v, \rho_2), \psi(v, \rho_1)]$, $\phi(r, \rho_1) < v$ and my claim follows as well. \square

All results for sector two follow easily from Lemmas 6, 7, 8, 10 and 11 as well as continuity of $\Delta_\rho P^2(\cdot)$. As Lemma 8 has an exact sector one analogue, the sector one results for $u \geq \max u^c$ follow from sector two results and Lemma 12. The results for $u < \max u^c$ follow from reasoning analogous to that in proof of Lemma 11 once we note that $\Delta_\rho M^2 \leq 0$ implies $\Delta_\rho M^1 \geq 0$. \square

To prove Propositions 3 and 6, it suffices to show that the impossibility property holds, as a (strict) increase in the spread of surplus implies a (strict) increase in vertical differentiation and the results follow from Theorem 2.

Proof of Proposition 3. The impossibility property is met, as $\Delta_\rho M^2 = 0$. \square

Proof of Proposition 6. Suppose the impossibility property does not hold, then $\Delta_\rho M^2 > 0$ and $\Delta_\rho v^c < 0$, which implies that $\Delta_\rho M^1 < 0$, $\Delta_\rho u^c \leq 0$ and trivially $\Delta_\rho P^2 < 0$ and $\Delta_\rho P^1 > 0$. Sector two expansion implies $M^2(\rho_1) < R^2$; sector one shrinkage implies $R^1 > M^1(\rho_2)$, and thus from (13)-(14) follows that:

$$\pi^1(u^c(\rho_2), P^2(\rho_2), \rho_2) \leq \pi^2(v^c(\rho_2), P^2(\rho_2)) \quad (33)$$

$$\pi^2(v^c(\rho_1), P^2(\rho_1)) \leq \pi^1(u^c(\rho_1), P^2(\rho_1), \rho_1). \quad (34)$$

Given that $\pi_v^2 > 0$ and $\pi_h^2 \geq 0$, we have that RHS of (33) is strictly less than the LHS of (34) and therefore $\pi^1(u^c(\rho_2), P^2(\rho_2), \rho_2) < \pi^1(u^c(\rho_1), P^2(\rho_1), \rho_1)$. However, as $\Delta_\rho \pi^1(u^c(\rho_1), P^1(\rho_1)) \geq 0$, $\pi_u^1 > 0$ and $\pi_h^1 \geq 0$ this is impossible and impossibility property holds. \square

Lemma 13. Scarce jobs and strict spread of surplus imply that $\Delta u^* \leq 0$, $\Delta v^* \geq 0$, with at least one of these holding strictly.

Proof. The first part follows trivially from Lemmas 8 and 12. Suppose $\Delta v^* = 0$; consider the set $\Omega^T = \{v \in [\max v^c, \min v^*] : \Delta\psi(v) < 0\}$ and its minimum v^5 . Suppose $v^5 \neq \min v^*$, then, by Theorem 2, $\Delta P^2(\min v^*) > 0$, which implies $\Delta v^* > 0$, contradiction. Therefore, if $\Delta v^* = 0$, then $\Delta\psi(\min v^*) < 0$, which implies $\Delta u^* < 0$ and concludes the proof. \square

Proof of Proposition 4. (i) From Theorem 2 in Appendix B and Corollary 1 follows that for (strictly) scarce jobs a strict spread results in a (strict) decrease in v^c . As C^2 remains unchanged and surplus function is supermodular, the (strict) increase in lowest wage follows from inspection of Equation (1). Note that for any $v'' > v' \geq v^c$ we have:

$$w^2(v'') = \int_{v'}^{v''} \pi_v^2(r, P^2(r))dr + w^2(v'). \quad (35)$$

As $P^2(r)$ increases and surplus is supermodular, it follows that $w^2(v'')$ increases by more than $w^2(v')$.

(ii) Proposition 3 and of Lemma 13 (in Appendix B) imply that $u^*(\rho_2) \leq u^*(\rho_1)$ and $v^*(\rho_2) \geq v^*(\rho_1)$ and at least one of these inequalities is strict. This gives:

$$\begin{aligned} w^1(u^*(\rho_1), \rho_2) &\geq w^1(u^*(\rho_2), \rho_2) = w^2(v^*(\rho_2), \rho_2) \geq w^2(v^*(\rho_1), \rho_2) \\ w^2(v^*(\rho_2), \rho_1) &\geq w^2(v^*(\rho_1), \rho_1) = w^1(u^*(\rho_1), \rho_1) \geq w^1(u^*(\rho_2), \rho_1) \end{aligned}$$

with at least one inequality holding strictly, which trivially implies:

$$w^1(u^*(\rho_1), \rho_2) - w^1(u^*(\rho_1), \rho_1) > w^2(v^*(\rho_2), \rho_2) - w^2(v^*(\rho_2), \rho_1). \quad (36)$$

Thus, $w^1(u^*(\rho_1))$ increases strictly. For any $u > u^*$ we have that:

$$w^1(u) = \int_{u^*}^u \pi_u^1(r, G^1(r))dr + w(u^*(\rho_1)). \quad (37)$$

For $u > u^*(\rho_1)$, $G^1(u)$ does not change; and as surplus' spread implies that $\pi_u^1(u, h)$ strictly increases, it follows that $w^1(u, \rho_2) > w^1(u, \rho_1)$ for any $u \in [u^*(\rho_1), 1]$.

(iii) Follows from (i), (ii) and the fact that with scarce jobs $C^i(\rho_2) = C^i(\rho_1)$.

I will turn now to wages of the least talented agents with strictly scarce jobs. By Theorem 2 we have that $u^c(\rho_2) > u^c(\rho_1)$. As wages strictly increase in talent, it follows from definition of critical ability that $w^1(u^c(\rho_2), \rho_1) > w^1(u^c(\rho_1), \rho_1) = 0$. Note that existence of a positive mass of agents for whom wages decrease (increase) follows from continuity of wage functions. \square

Proof of Proposition 5. The first claim follows from inspection of Equation (38): each firm is matched with a less productive agent, so $(P^2)^{-1}$ decreases for all h and the profit constant falls as well, as it is equal to $\pi^2(v^c, 0)$ and v^c falls by Proposition 3. The second follows trivially from the fact that the pool of sector two firms is unchanged, the supply of talent falls and surplus is increasing in talent.

For simultaneous increases in surplus' spread and levels, the improvement in skills for $u \geq u^c(\rho_2)$ implies that $\pi_h^2(u, h)$ increases for all $(u, h) \in [u^c(\rho_1), 1] \times [0, 1]$; as $(P^1)^{-1}(u)$ and C_R^1 increase by Proposition 3 and Lemma 1, the result wrt sector one profits follows from inspection of Equation (38) as well.

As for the last two claims, note that, fixing the assignment, an increase in surplus' levels increases the total output in the economy. As the stable assignment is surplus maximising in this model, the change from the old to new stable assignment has to further improve total output⁵⁰. Finally, as sector two's total output falls, it has to increase in sector one. \square

Lemma 14. If jobs are abundant and surplus both increases and becomes strictly more spread out, the lowest profit rises in sector one and a decreases in sector two ($C_R^1(\rho_2) \geq C_R^1(\rho_1)$ and $C_R^2(\rho_2) \leq C_R^2(\rho_1)$).

Proof. In sector one, there are two possibilities: $M^1(\rho_1) < R^1$ and $M^1(\rho_1) = R^1$. If the former is the case, then $C_R^1(\rho_1) = 0$ and the result follows trivially. If the latter is true, then $M^1(\rho_2) = R^1$ and by Proposition 6 we have $u^c(\rho_2) \geq u^c(\rho_1)$, $v^c(\rho_2) \leq v^c(\rho_1)$, $P^1(\rho_1) = P^2(\rho_2)$ and $P^2(\rho_2) = P^2(\rho_1)$. This implies that $C_R^1(\rho_2) \geq C_R^1(\rho_1)$, as by measure consistency and Lemma 1:

$$C_R^1(\rho_i) = \pi^1(u^c(\rho_i), P^1(\rho_i)) - \pi^2(v^c(\rho_i), P^2(\rho_i)),$$

for $i = 1, 2$. The result for sector two follows from analogous reasoning, but the two cases are $M^2(\rho_2) < R^2$ and $M^2(\rho_2) = R^2$. \square

⁵⁰This is the case, as my model can be rewritten as a special case of the assignment model described in Gretskey et al. (1992) and thus the equivalence of stable and efficient matching showed by them holds for my model as well.

Proof of Proposition 7. Consider $T = A_A^1(\rho_2) \cap A_A^1(\rho_1)$, the set of agents who work in sector two in both matching problems. Denote the least talented of those agents $-\inf_y T -$ as $\max_\rho v^c$. Her wage depends on two factors: positively on the surplus she produces and negatively on its share received by the firm she is matched with. The first factor always increases, as she is matched with a more productive firms. The change in the second factor can be both positive (for $\max_\rho v^c = v^c(\rho_1)$) and negative (for $\max_\rho v^c = v^c(\rho_2)$). If the former is the case, however, then the increase in surplus received by her firm:

$$\Delta_\rho r^2(P^2(v^c(\rho_1))) = \int_{v^c(\rho_2)}^{v^c(\rho_1)} P_v^2(r, \rho_2) \pi_h^2(r, P^2(r, \rho_2)) dr + \Delta_\rho C_R^2,$$

is always less than the increase in the surplus she produces (by Lemma 14):

$$\begin{aligned} \pi^2(v^c(\rho_1), P^2(v^c(\rho_1), \rho_2)) - \pi^2(v^c(\rho_1), P^2(\rho_1)) \\ = \int_{v^c(\rho_2)}^{v^c(\rho_1)} P_v^2(r, \rho_2) \pi_h^2(v^c(\rho_1), P^2(r, \rho_2)) dr. \end{aligned}$$

Thus, $w^2(\max_\rho v^c, \rho_2) - w^2(\max_\rho v^c, \rho_1) \geq 0$ and by inspection of Equation (35) we have that $w^2(v'', \rho_2) - w^2(v'', \rho_1) \geq w^2(v', \rho_2) - w^2(v', \rho_1)$, for any $v'' > v' \geq \max_\rho v^c$. It follows that wages increase for all $v \in T$. This and revealed preference imply that all agents who used to work in sector two are better off⁵¹. As the top wages increase by more in sector one (by the same reasoning as in the proof of Proposition 4) it follows that wages increase for most talented sector one workers. \square

Proof of Proposition 8. The result wrt sector one (two) profits follows from Proposition 6, the definition of an increase (decrease) in supply, the definition of PAM ($P^i(\cdot)$), Lemma 14 in Appendix B and inspection of Equation (38). The increases in total output follow from analogous reasoning as in the proof of Proposition 5. \square

Proof of Proposition 9. I will prove it for sector one, results for sector two follow from symmetry. (i) As all agents work in one of the sectors or are unmatched, it follows from measure consistency and symmetry that for any $u \geq u^c$ we have:

$$2M^1 G^1(u) + \max\{1 - 2R^1, 0\} = C(u, u).$$

⁵¹This is trivial if $v^c(\rho_2) \leq v^c(\rho_1)$. If $v^c(\rho_2) \geq v^c(\rho_1)$ then the agents with $v \in [v^c(\rho_1), v^c(\rho_2))$ will move to sector one; but as the lowest wages are the same in both sectors, they earn more than $w^2(v^c(\rho_2), \rho_2)$, which in turn is greater than their old wage.

In a symmetric problem, sector sizes depend only on R^1 , which implies that:

$$2M^1(\rho_1)(G^1(u, \rho_2) - G^1(u, \rho_1)) = C(u, u, \rho_2) - C(u, u, \rho_1).$$

To complete the proof, it suffices to note that the definition of concordance implies that RHS is positive and that $u^c(\rho_2) \leq u^c(\rho_1)$ by Corollary 3.

(ii) Recall Equation (2). The wage constant, C^1 , remains unchanged⁵². Since surplus function, marginals and sector sizes do not change either, by Definition 6 wages depend only on within-sector talent distributions and hence, they increase for any $u > u^c(\rho_2)$; for $u \in [0, u^c(\rho_2)]$ they remain constant (and equal to 0).

(iii) Trivial, as $C^1(\rho_2) = C^1(\rho_1)$ (see footnote 52) and $w^1(1, \rho_2) \geq w^1(1, \rho_1)$.

(iv) Denote the worst matched firm in sector i as $P^i = \frac{R^i - M^i}{R^i}$. As the sector size does not change, neither do P^i and A_F^i (the set of matched firms)⁵³. All unmatched firms make zero profit and the profit function for all matched firms in sector i is given by (see Sattinger, 1979):

$$r^i(h) = \int_{P^i}^h \pi_h^i((P^i)^{-1}(r), r) dr + C_P^i. \quad (38)$$

The profit constant will either remain unchanged (for $R^1 \geq \frac{1}{2}$) or will fall (for $R^1 < \frac{1}{2}$, as then $C_P^i = \pi^1(u_c, 0)$ and u^c decreased). Thus, (iv) follows from (i).

(v) By (i), every firm produces lower surplus. \square

C Supply and Demand: Definitions and Shifts

In order to recover the assignment from the sectoral wage functions we need to extend the latter to the full talent support. Note that if an agent with $u < u^c$ wanted to join sector one, she would need to compensate the worst firm in that sector for the lost surplus and hence her wage would be $w^1(u^c) - (\pi^1(u, P^1) - \pi^1(u^c, P^1))$. Given this we can define the extended sectoral wage functions:

$$w^{1e}(u) = \begin{cases} w^1(u^c) - (\pi^1(u, P^1) - \pi^1(u^c, P^1)), & u < u^c, \\ w^1(u), & u \in [u^c, u^*], \end{cases}$$

⁵²If jobs are scarce ($R^1 \leq \frac{1}{2}$), then $C^1 = 0$. If jobs are abundant ($R^1 > \frac{1}{2}$), then $C^1 = \pi^1(u^c, M^1)$ and by Corollary 3 we have that $u^c = 0$.

⁵³For non-strictly supermodular surplus functions and the abundant jobs case ($R^1 + R^2 > 1$), the set of matched firms is not unique. Hereafter, I will assume that in such cases firms with $h \geq P^i$ become matched. This simplifies notation and is without any loss in generality, as the profits of the relevant firms will be 0.

$$w^{2e}(v) = \begin{cases} w^2(v^c) - (\pi^2(v, P^2) - \pi^2(v^c, P^2)), & v < v^c, \\ w^2(v), & v \in [v^c, v^*]. \end{cases}$$

Hence, the overall wage function $w(\bullet)$ is given by $w(u, v) = \max\{w^{1e}(u), w^{2e}(v), 0\}$.

Definition 16. Given a wage function $w(\bullet)$, the *cumulative supply of talent* of talent t in sector 1 is $S^1(t) = \Pr(w^{1e}(U) \geq w^{2e}(V), w^{1e}(U) \geq 0, U \geq t)$ and in sector 2: $S^2(t) = \Pr(w^{2e}(V) > w^{1e}(U), w^{2e}(V) \geq 0, V \geq t)$.

Note that for the equilibrium sectoral wage functions we have by Equation 5 and Lemma 3 we have that $S^i(t) = M^i(1 - G^i(t))$ from which Definition 9 follows.

Definition 17. Talent supply *shifts up* in sector one if, given the old equilibrium wage function $w(\bullet, \rho_1)$, $S^1(t, \rho_2) \geq S^1(t, \rho_1)$ for all t . It shifts up strictly if this holds strictly for some t .

The definitions of talent demand and its shift are more complicated, as talent demand – unlike supply – is not necessarily unique in this model.

Definition 18. A mapping $I : [0, 1] \rightarrow [0, 1] \cup \{-1\}$ is a sector one *input function* for wage function $w^1(\cdot)$, if a) for $I(h) \in [0, 1]$, $I(h) \in \arg \max_u \pi^1(u, h) - w^1(u)$ and $\pi^1(I(h), h) - w^1(I(h)) \geq 0$ and b) for $I(h) = -1$, $\pi^1(u, h) - w^1(u) < 0$ for all $u \in [0, 1]$.

Given a talent level u and an input function $I(\cdot)$, define the set $B(u, I) = \{h \in [0, 1], I(h) \geq u\}$.

Definition 19. A mapping $D_C : [0, 1] \rightarrow [0, R]$ is a sector one *cumulative talent demand function* for wage function $w^1(\cdot)$, if there exists an input function such that $R^1 \int_{B(u, I)} 1 du = D_C(u)$, for all $u \in [0, 1]$.

For any matching problem, I will denote as $DC(\rho)$ the set of all possible cumulative demand functions and as $DC(u, \rho)$ the set of their values for talent u .

Definition 20. Sector one talent demand *shifts up* if $\inf DC(u, \rho_2) \geq \sup DC(u, \rho_1)$, for the old equilibrium wage function $w^1(\cdot, \rho_1)$ and all $u \in [0, 1]$.

Proposition 10. Suppose sector one surplus is strictly supermodular. Then a simultaneous increase in surplus' spread and levels shifts demand for talent up. If jobs are scarce, a spread of surplus alone suffices for an upward shift of talent demand.

Proof. For any h and any input function $I(\cdot)$ it has to be the case that either $I(h) = -1$ or $I(h) = 1$ or $w_u(I(h)) \geq \pi_u^1(I(h), h)$ (otherwise the firm would demand an agent of higher talent). For each h , denote the minimum of the set of all $I(h)$'s as $\underline{I}(h)$ and its maximum as $\bar{I}(h)$, note that they both exist and define $\Omega_I = \{h : \underline{I}(h) \neq -1\}$.

Lemma 15. An increase in surplus' spread implies $\underline{I}(h, \rho_2) \geq \bar{I}(h, \rho_1)$ for all $h \in \Omega_I(\rho_1) \cap \Omega_I(\rho_2)$.

Proof. If $\underline{I}(h, \rho_2) = 1$, the claim holds. Suppose $\underline{I}(h, \rho_2) \neq 1$ then we have

$$\pi_u^1(\underline{I}(h, \rho_2), P^1(\underline{I}(h, \rho_2), \rho_1), \rho_1) \geq \pi_u^1(\underline{I}(h, \rho_2), h, \rho_2).$$

Note that the strict increase in spread implies that $\Delta_\rho \pi^1(u, h) > 0$ for any (u, h) and, hence, it has to be the case that $P^1(\underline{I}(h, \rho_2), \rho_1) > h = P^1(\bar{I}(h, \rho_1))$ (by the fact that, under strict supermodularity, the unique stable matching is PAM). \square

From this, the second claim follows readily, as scarcity of jobs implies $\Omega_I(\rho_1) = \Omega_I(\rho_2)$. As for the first claim, note that the increase in surplus levels means that each firm's profit increases for the old choice of inputs, and hence, by profit maximisation, also for the new choice. Thus, no firms leave the market and the result follows. \square

References

- Anderson, S. (2003). Why Dowry Payments Declined with Modernization in Europe but Are Rising in India. *Journal of Political Economy* 111(2), pp. 269–310.
- Becker, G. S. (1973, July-Aug.). A Theory of Marriage: Part I. *Journal of Political Economy* 81(4), 813–46.
- Bickel, P. and E. Lehmann (1979). Descriptive Statistics for Nonparametric Models IV. Spread. In J. Jureckova (Ed.), *Contributions to Statistics : Jaroslav Hajek Memorial Volume*. D. Reidel Pub. Co Dordrecht, Holland ; Boston.
- Bielecki, A. (1956). Une remarque sur la methode de Banach-Cacciopoli-Tikhonov dans la theorie des equations differentielles ordinaires. *Bull. Acad. Polon. Sci. Cl. III. 4*, 261–264.
- Bond, P. and V. Glode (2014). The labor market for bankers and regulators. *Review of Financial Studies*.

- Chiappori, P.-A., R. McCann, and L. Nesheim (2010, February). Hedonic price equilibria, stable matching, and optimal transport: equivalence, topology, and uniqueness. *Economic Theory* 42(2), 317–354.
- Chiappori, P.-A., S. Oreffice, and C. Quintana-Domeque (2011). Matching with a Handicap: The Case of Smoking in the Marriage Market. (1011-07).
- Chiappori, P.-A., S. Oreffice, and C. Quintana-Domeque (2012). Fatter Attraction: Anthropometric and Socioeconomic Matching on the Marriage Market. *Journal of Political Economy* 120(4), 659 – 695.
- Costinot, A. and J. Vogel (2010, 08). Matching and Inequality in the World Economy. *Journal of Political Economy* 118(4), 747–786.
- Costrell, R. M. and G. C. Loury (2004, December). Distribution of Ability and Earnings in a Hierarchical Job Assignment Model. *Journal of Political Economy* 112(6), 1322–1363.
- Dupuy, A. (2015, January). The Assignment of Workers to Tasks with Endogenous Supply of Skills. *Economica*.
- Gabaix, X. and A. Landier (2008, February). Why Has CEO Pay Increased so Much? *The Quarterly Journal of Economics* 123(1), 49–100.
- Gola, P. (2015). Occupational Sorting and the Structure of Status. University of Oxford. PDF file.
- Gretsky, N. E., J. M. Ostroy, and W. R. Zame (1992, January). The Nonatomic Assignment Model. *Economic Theory* 2(1), 103–27.
- Grossman, G. M., E. Helpman, and P. Kircher (2013, October). Matching and Sorting in a Global Economy. NBER Working Papers 19513, National Bureau of Economic Research, Inc.
- Hasselblatt, B. and A. Katok (2003). *A First Course in Dynamics: With a Panorama of Recent Developments*. Cambridge University Press.
- Legros, P. and A. F. Newman (2002). Monotone matching in perfect and imperfect worlds. *Review of Economic Studies* 69, 925–942.
- Lindenlaub, I. (2014). Sorting Multidimensional Types: Theory and Application. European University Institute, Florence. PDF File.

- Lucca, D., A. Seru, and F. Trebbi (2014, June). The revolving door and worker flows in banking regulation. Working Paper 20241, National Bureau of Economic Research.
- McCann, R. J., X. Shi, A. Siow, and R. Wolthoff (2015). Becker meets ricardo: Multisector matching with communication and cognitive skills. *Journal of Law, Economics, and Organization* 31(4), 690–720.
- Melitz, M. J. (2003, November). The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity. *Econometrica* 71(6), 1695–1725.
- Rosen, S. (1981). The economics of superstars. *American Economic Review* 71(5), 845–58.
- Roy, A. D. (1951). Some Thoughts On The Distribution Of Earnings. *Oxford Economic Papers* 3(2), 135–146.
- Sampson, T. (2014). Selection into Trade and Wage Inequality. *American Economic Journal: Microeconomics* 6(3), 157–202.
- Sattinger, M. (1975). Comparative advantage and the distributions of earnings and abilities. *Econometrica* 43(3), 455–468.
- Sattinger, M. (1979, March). Differential Rents and the Distribution of Earnings. *Oxford Economic Papers* 31(1), 60–71.
- Sattinger, M. (1993). Assignment Models of the Distribution of Earnings. *Journal of Economic Literature* 31(2), pp. 831–880.
- Scarsini, M. (1984). On measures of concordance. *Stochastica* 8(3), 201–218.
- Shaked, M. and G. Shanthikumar (2007). *Stochastic Orders* (1 ed.). Springer Series in Statistics. Springer-Verlag New York.
- Shapley, L. and M. Shubik (1971). The Assignment Game I: The Core. *International Journal of Game Theory* 1(1), 111–130.
- Tervio, M. (2008, June). The Difference That CEOs Make: An Assignment Model Approach. *The American Economic Review* 98(3), 642–668.
- Teulings, C. (2005). Comparative Advantage, Relative Wages, and the Accumulation of Human Capital. *Journal of Political Economy* 113(2), 425–461.

Teulings, C. N. (1995, April). The Wage Distribution in a Model of the Assignment of Skills to Jobs. *Journal of Political Economy* 103(2), 280–315.